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PHYSICS

Mechanics of material point
Destined for students of first year Science of matter

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Contents

Preface

Mechanics is a fundamental science that is essential for understanding the physical world by developing theories based on experiments. This course includes the study of mechanics of a material point where the concept of material point is a powerful tool for modeling the motion of objects in classical mechanics. By ignoring the spatial dimensions of objects, material points can be used to simplify complex problems and make them easy to understand.

This course is destined for students in the first year of the matter science (SM). It is composed of four chapters followed by an appendix which allows the student to understand the mechanics of the material point.

The first chapter explores into the definitions of fundamental physical quantities like units and dimensions in physics, vectors and coordinate systems, providing the groundwork for understanding and solving mechanics problems at the university level and allows the student to become familiar with the language and technical words used in physics.

In the second chapter we were interested in the study of kinematics, which consists in the study of the motion of a body (the material point) without approaching the causes which produce this motion. The determination of the equation of the trajectory traveled by the body, the equation of motion, the speed and the acceleration of the body are discussed in this chapter.

In the third chapter we were interested in the dynamics of the material point which is the science which treats the relations between the forces applied on the body in order to describe the movement.

The fourth chapter is devoted to the course on work and energy to solve dynamic problems using the notions of work forces, kinetic energy and potential energy.

Finally, this course ends with two appendixes, the first, devoted mathematical reminder on differential operators and the second includes terminological equivalents to English of all terms and basic scientific concepts to facilitate understanding of previous chapters.

Chapter I:

Units, vectors and coordinate systems

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I.1 Units and measurements

I.1.1 Basic units of measurement

Physical quantity: the amount or number (value) of a material or a system that is measured (e.g. mass, length, time, volume, pressure, temperature)

Unit of measurement: is a standard quantity defines the magnitude of a physical quantity

Examples: (meter is the base unit of length; gram is the base unit of mass)

There were many different systems of units in use around the world. Some of the most common ones were: the metric system (used in the United Kingdom and some other countries), the CGS system (based on the centimeter, gram, and second), the FPS system (based on the foot, pound, and second).

In 1960 an international committee agreed on a standard system of units for the fundamental quantities of science, which is the modern form of the metric system, called International System of Units (for the French, Système International d'unités) and is abbreviated as (SI). It is based on seven base units:

Meter (m): the unit of length

Kilogram (kg): the unit of mass

Second (s): the unit of time

Ampere (A): the unit of electric current

Kelvin (K): the unit of thermodynamic temperature

Mole (mole): the unit of amount of substance

Candela (cd): the unit of luminous intensity

The SI system also includes a number of derived units, for example, the unit of speed is the meter per second (m/s), the unit of area is square meter (m^2), the unit of acceleration is meter per Second Square (ms^{-2}), the unit of force is Newton (N) ($kgms^{-2}$), and the unit of Frequency is the Hertz (Hz) (s^{-1}) etc.

I.1.2 Dimensions of physical quantity

The dimension of a physical quantity denotes the physical nature of a quantity; it used to define the unity of length, of mass, of time etc.

The distance between two points can be measured (dimension) by meter, by feet, by inches, by yard or by miles. For the SI the unit for the dimension of length is meter. For that, a fundamental (base) set of dimensions was chose and the SI units were defined for those dimensions.

The following table shows the dimension of the seven units of the SI system:

Quantity	Symbol	Dimension	Si base unit
Length	l	L	meter (m)
Mass	m	M	kilogram (kg)
Time	t	T	second (s)
Electric Current	I	I	ampere (A)
Temperature	T	Θ	kelvin (K)
Amount of substance	n	N	mole (mol)
Luminous intensity	I_v	J	candela (cd)

To denote a dimension of a physical quantity X, we use square brackets around the symbol [X]. For example, if r is the radius of a cylinder and h is its height, we write [r] = L and [h] = L to indicate the dimensions of the radius and height.

I.1.3 Dimensional analysis

From the base dimensions of the base SI units, we can derive any other dimension by using **dimensional analysis**. That require using the dimensions of physical quantities to check the consistency of equations. We can add or subtract two quantities if they have the same dimension, we can also use multiplication and division of dimensions.

For example, the dimension of speed is the dimensions length [L] over time [T]: $[v] = \frac{L}{T} = L.T^{-1}$,

The dimension of acceleration is the dimensions of length divided by time squared: $[a] = \frac{L}{T^2} = L.T^{-2}$

In the general case, the dimension equation of any quantity [X] can be write in the form:

$[X] = L^a.M^b.T^c.I^d.\Theta^e.J^f.N^g$

Where, the letters M, L, T, I, Θ, J and N represent the dimensions of mass, length, time, electric current, temperature, luminous intensity and amount of substance, respectively. The exponents a, b, …,g are positive or negative integers that tell us how many times each dimension is multiplied together to get the dimension of X.

Other examples of the dimensions of different physical quantities:

Energy: $ML^2\,T^{-2}$

Power: $ML^2\,T^{-3}$

Work: $ML^2\,T^{-2}$

There are physical quantities do not possess any dimension like angular displacement (radian), solid angle (Steradian), refractive index etc.

Examples:

1/ The unit of force is: $F = ma = kg.m.s^{-2} = N$

The dimension of force is: $[F] = [m].[a] = M.\frac{L}{T^2} = M.L.T^{-2}$

2/ The force of attraction between two masses m_1 and m_2 is given by Newton's law:

$\vec{F}_{1\to2} = -G\frac{m_1.m_2}{r^2}\vec{u}_{1\to2}$

We can conclude the unit of the Gravitation constant G: $G = \frac{F.r^2}{m_1.m_2}$

$$[G] = \frac{[F].[r^2]}{[m_1].[m_2]} = \frac{M.L.T^{-2}.L^2}{M.M} = \frac{L^3.T^{-2}}{M} = M^{-1}.L^3.T^{-2}$$

So, the unit of G is: $kg^{-1}.m^3.s^{-2}$

I.2 Mathematical reminder

I.2.1 Derivatives

Differentiation is finding the instantaneous rate of change of a function with respect to its variable. Differentiation is symbolized by several notation s:

If the function is f(x), then the derivative of the function with respect to x is written in the form:

- ✓ $\frac{df}{dx}$ Leibniz's notation
- ✓ $f'(x)$ Lagrange's notation
- ✓ $\dot{f}(x)$ Newton's notation

The most widely used notation is Leibniz's notation.

If the function has a lot of variables (exp: $\boldsymbol{f(x,y,z)}$) then we use the partial derivatives: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

In this case, if we want to calculate the derivative of the function with respect to x, we consider y and z constants.

Example

The speed is the derivative of distance with respect to time $\boldsymbol{v(t)} = \boldsymbol{\frac{dx}{dt}}$

If the equation of motion is: $\boldsymbol{x(t) = 2t^2 + 1}$

The speed is: $\boldsymbol{v(t)} = \boldsymbol{\frac{dx}{dt}} = \boldsymbol{4t}$

The acceleration is the derivative of speed with respect to time: $a(\boldsymbol{t}) = \boldsymbol{\frac{dv}{dt}} = \boldsymbol{4}$

If we have an equation with two variables, we can differentiate both sides of the equation:

Example

Given the equation: $y = 4\mathrm{t}^2 - 2\mathrm{t}$

We derive the first side: $\frac{\mathrm{d(y)}}{\mathrm{dy}} = 1$

We derive the second side: $\frac{\mathrm{d(4t^2-2t)}}{\mathrm{dt}} = 8\mathrm{t} - 2$

And we write: $1.dy = (8t - 2)dt \iff dy = (8t - 2)dt$

Some famous derivatives

F(x) and g(x) two functions and (a) is a number (constant):

- $(f+g)' = \frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$
- $(f.g)' = \frac{d(f.g)}{dx} = \frac{df}{dx}.g + \frac{dg}{dx}.f$
- $\left(\frac{f}{g}\right)' = \frac{\frac{df}{dx}.g - \frac{dg}{dx}.f}{g^2}$
- $\left(\frac{1}{f}\right)' = -\frac{\frac{df}{dx}}{f^2}$
- $\left(\sqrt{f}\right)' = \frac{\frac{df}{dx}}{2\sqrt{x}}$
- $((f)^n)' = (n-1)f'.f^{n-1}$
- $\frac{da}{dx} = 0$
- $\frac{d(fa)}{dx} = a.\frac{df}{dx}$
- $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}e^x = e^x$

Derivatives of a composite function

A composite function is a function is written in terms of another function for example $f(g)$ is a function with respect to a second function g, while g is a function with respect to x and we write $(f\ o\ g)(x) = f(g(x))$

To derivative a composite function, we differentiate the first function with respect to the second function and then differentiate the second function with respect to the variable

$$(f\ o\ g)'(x) = f'\big(g(x)\big).g'(x)$$

$$\boldsymbol{\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}}$$

Examples

1/ given the function $f(\theta) = \cos\theta\ ;\ \theta = at$

The function f *have the variable* θ *while* θ *have the variable* t.

The derivate the function f *with respect to* t *is:* $\frac{df(\theta)}{dt} = \frac{df(\theta)}{d\theta}.\frac{d\theta(t)}{dt}$

We have: $\frac{df(\theta)}{d\theta} = \frac{d\cos\theta}{d\theta} = -\sin\theta\ \ ;\frac{d\theta(t)}{dt} = \frac{d(at)}{dt} = a$

So: $\frac{df(\theta)}{dt} = (-\sin\theta).a = -a\sin\theta$

2/ given the function $y(x) = \cos\theta$ *with* $\theta = x^2$

$$\frac{dy}{dx} = \frac{d\cos\theta}{d\theta}.\frac{d\theta}{dx}$$

$$\frac{d\cos\theta}{d\theta} = -\sin\theta \quad ; \quad \frac{d\theta}{dx} = \frac{d(x^2)}{dx} = 2x$$

$$\frac{dy}{dx} = -2x.\sin x^2$$

Derivative of trigonometric functions

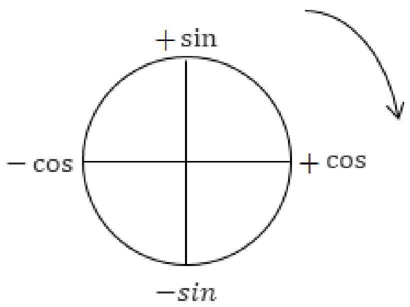
To derivate the basic trigonometric functions: sine ($\sin x$), cosine ($\cos x$)

To differentiate the basic trigonometric functions cosine (cos) and sine (sin) we can use the trigonometric circle to arise the two derivatives.

We follow the clockwise direction to find the derivative:

$$\frac{d}{dx}(\sin x) = \cos x \quad ; \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(-\sin x) = -\cos x \quad ; \frac{d}{dx}(-\cos x) = +\sin x$$



The inverse function of the $cosx$ function is arccos , we use it to find the angle x.

Examples

$cos\,x = 0{,}5 \Rightarrow x = arccos\,(0{,}5) = 1{,}047\,rad = 60°$

$sin\,x = 0{,}5 \Rightarrow x = arcsin\,(0{,}5) = 0{,}523\,rad = 30°$

I.2.2 Integrals

Integration is the inverse process of differentiation, and we denote the integration of the function $\boldsymbol{f(x)}$ with respect to the variable x by: $\int f(x)dx$

If the integration is definite between two values x_1 and x_2, then the integration is written: $\int_{x_1}^{x_2} f(x)dx$, so we integrate and put a line $\Big|_{x_1}^{x_2}$ and then substitute the value x_1 and x_2.

If the integration is indefinite, after integration we add a constant C to the obtained result.

On the basis that integration is the opposite of differentiation, then: $\int x^n dx = \frac{1}{n+1}.x^{n+1} + C$

Example

The integral of the function $f(x) = x^3$ between $x_1 = 0$ and $x_2 = 2$ is:

$$\int_{x_1}^{x_2} x^3 dx = \int_0^2 x^3 dx = \frac{1}{4}x^4 \Big|_0^2 = \frac{1}{4}(2^4 - 0^4) = \frac{1}{4}.16 = 4$$

Some famous integrals

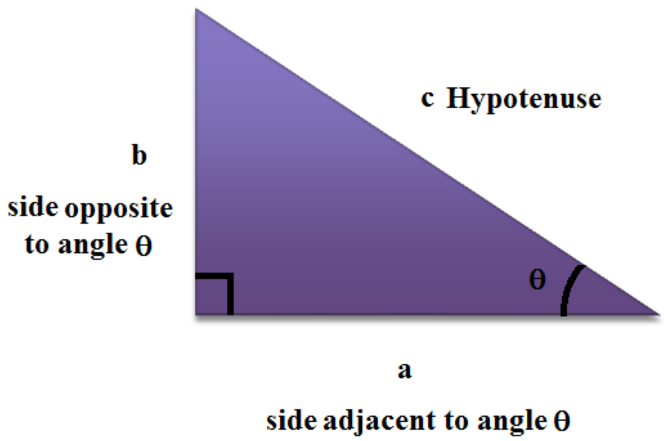
f(x) and g(x) two functions, a in an integer number,

- $\int (a.f)dx = a\int f dx$
- $\int (f+g)dx = \int f dx + \int g dx$
- $\int \frac{1}{x} dx = ln\,x$
- $\int \frac{dx}{a+bx} dx = \frac{1}{b} ln(a+bx)$
- $\int e^{ax} dx = \frac{1}{a} e^{ax}$
- $\int sin\,x\,dx = -cos\,x$
- $\int cos\,x\,dx = sin\,x$

I.2.2 Pythagorean theorem

In a right triangle we can write: $\boldsymbol{c^2 = a^2 + b^2}$

a is the adjacent of the angle θ, b is the side opposite of the angle θ and c is the hypotenuse.



We have also: $\boldsymbol{\cos\theta = \frac{a}{c}}$; $\boldsymbol{\sin\theta = \frac{b}{c}}$; $\boldsymbol{tg\,\theta = \frac{b}{a}}$

I.2 Scalar and vector quantities

In physics, there are two types of quantities, scalar quantities and vector quantities.

I.2.1 Scalar quantities

A scalar is a physical quantity that always defined by a number (numerical value) associated by an appropriate unit for example mass of a body ($5\ Kg$), temperature ($20\ °C$), volume ($3\ m^3$), area ($2\ m^2$), etc.

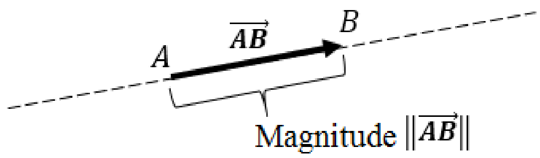
I.2.2 Vector quantities

A vector is a physical quantity that is represented graphically by an arrow. This arrow is an oriented segment with a starting point (A) (the "tail") (point of application of the vector) and an ending point (B) (the "head").

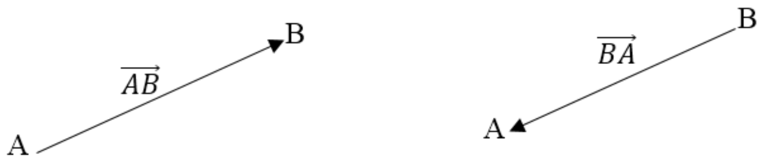
A vector can denoted by the two points (the starting and the ending points) AB with an arrow above $\overrightarrow{\boldsymbol{AB}}$, by only a letter $\vec{\boldsymbol{V}}, \vec{\boldsymbol{A}}, \vec{\boldsymbol{B}}$ etc., or by a letter shows the indication of the quantity, for example: $\vec{v}$ for velocity, $\vec{a}$ for acceleration, $\vec{F}$ for force , etc.

A vector is characterized by:

Magnitude: is the length of the arrow, denoted $\|\overrightarrow{\boldsymbol{AB}}\|$, refers to the numerical value (scalar) or the intensity of a physical quantity. For example, the magnitude of a force is the amount of the applied force, e.g. $\|\vec{F}\| = 5$ Newton.



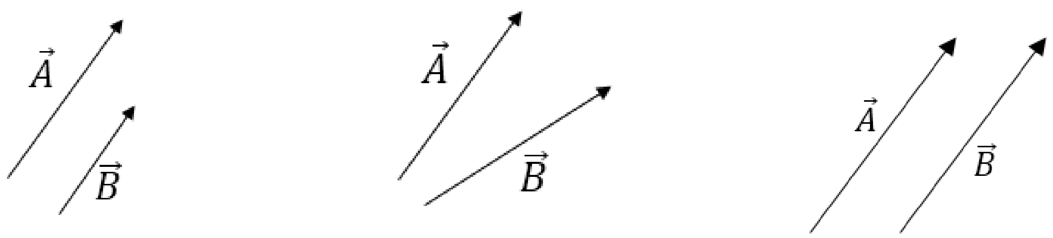
Sense: the sense of a vector is determined by its the orientation of its arrow; it is specified by the order of the starting and the ending points.



The vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have an opposite direction and a same magnitude, we can write: $\overrightarrow{AB} = -\overrightarrow{BA}$.

Direction: the direction of a vector refers to the angle between the vector and a reference axis or plane. We say that the direction of the weight of an object $\vec{P}$, is always vertical from top to bottom takes (Ox) the horizontal axe.

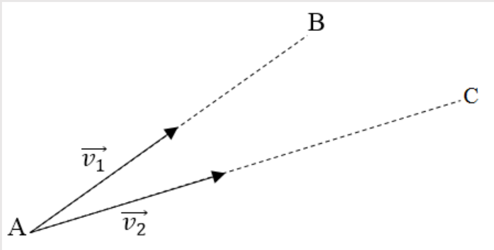
Two vectors are equal if they have the same direction and the same magnitude.



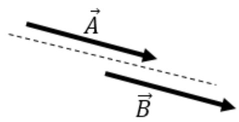
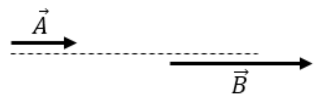
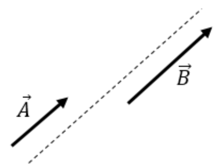
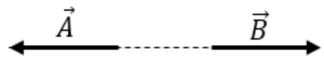
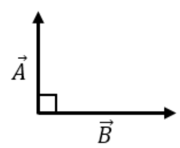
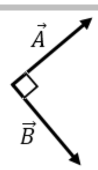
Vector $\vec{A}$ and vector $\vec{B}$ have same direction but different magnitude.	Vector $\vec{A}$ and vector $\vec{B}$ have same magnitude but different direction.	Vector $\vec{A}$ and vector $\vec{B}$ have same direction and same magnitude.

Example

Two cars (car 1 and car 2) having the same starting point A, each car have a velocity vector $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. The two velocities have not the same direction. With displacement, the car 1 will arrive to B point but the car 2 will arrive to point C (figure).



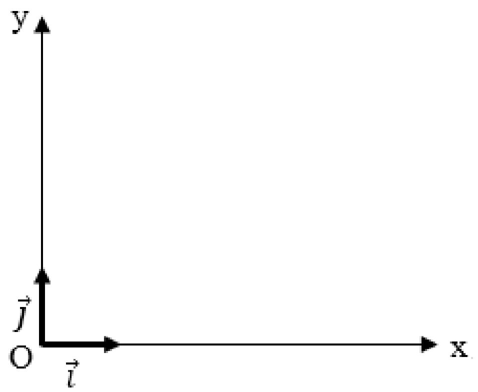
Various relations between two vectors

In these three cases: ❖ $\vec{A}$ is equal to $\vec{B}$: $\vec{A} = \vec{B}$ Because they are parallel and have identical magnitudes: $\|\vec{A}\| = \|\vec{B}\|$	  
❖ $\vec{A}$ is parallel to $\vec{B}$ but they are not equal because $\|\vec{A}\| \neq \|\vec{B}\|$	 
❖ $\vec{A}$ is antiparallel to $\vec{B}$ ($\vec{A} \neq \vec{B}$) because they are parallel but $\|\vec{A}\| = \|\vec{B}\|$	
❖ $\vec{A}$ is orthogonal to $\vec{B}$ ($\vec{A} \neq \vec{B}$)	 

I.3 Components of a vector in Cartesian coordinate system

I.3.1 Two-dimensional space

Two-dimensional space is characterized by an ordered pair of perpendicular axes (Oxy), origin point O and the unit vectors $(\vec{\imath},\vec{\jmath})$ of a single unit of length for both axes $\|\vec{\imath}\| = \|\vec{\jmath}\| = 1$.



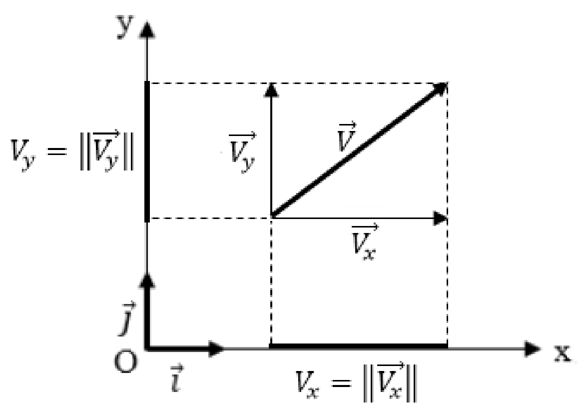
A vector $\vec{V}$ in two-dimensional space is described by a pair of its **vector components**: $\vec{V} = \vec{V_x} + \vec{V_y}$

$\vec{V_x}$ the x-coordinate called the x component is the orthogonal projection of the vector $\vec{V}$ onto Ox axis.

$\vec{V_y}$ the y-coordinate called the y component is the orthogonal projection of the vector $\vec{V}$ onto Oy axis.

$\vec{V_x}$ is written in one-dimensional space Ox as: $\vec{V_x} = V_x\vec{\imath}$, $\vec{V_y}$ is written in one-dimensional space Oy as: $\vec{V_y} = V_y\vec{\jmath}$.

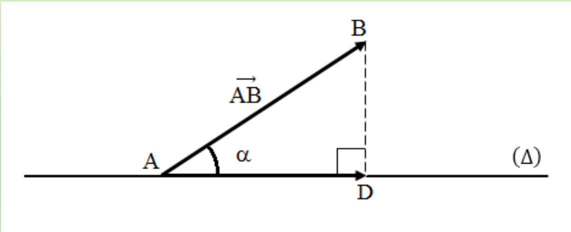
$V_x = \|\vec{V_x}\|$ and $V_y = \|\vec{V_y}\|$ are the **scalar components** of the vector $\vec{V}$ and we write: $\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath}$



Note:

The orthogonal projection of a vector $\overrightarrow{AB}$ on a line (Δ), making an angle α between them, is the vector $\overrightarrow{AD}$, where D is the projection of the point B on the line (Δ).

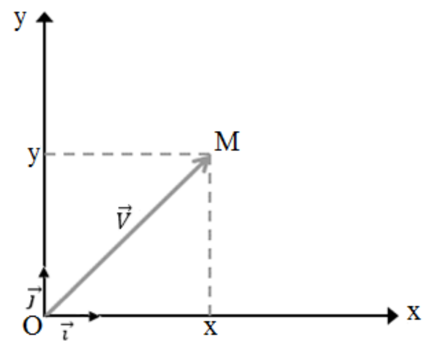
The relation between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is: $\|\overrightarrow{AD}\| = \|\overrightarrow{AB}\|.\cos\alpha$



I.3.1.2 Standard position vector

The standard position vector is the vector with its initial point at the origin of the plane O. In this case, its beginning point is $O(0,0)$ and its ending point is the point M with components M(x,y). The vector can be written as: $\vec{V} = x\vec{\imath} + y\vec{\jmath}$

The components of the vector are $\vec{V}\begin{pmatrix} x \\ y \end{pmatrix}$ and the components of the point $M(x, y)$.



If we know the coordinates of the starting and the ending points of the vector, we can obtain the scalar components of the vector.

Consider a vector $\vec{V}$, it's starting point is $A(x_A, y_A)$ and it's ending point is $B(x_B, y_B)$.

Scalar components of the vector $\vec{V}$ are: $\begin{cases} V_x = x_B - x_A \\ V_y = y_B - y_A \end{cases}$

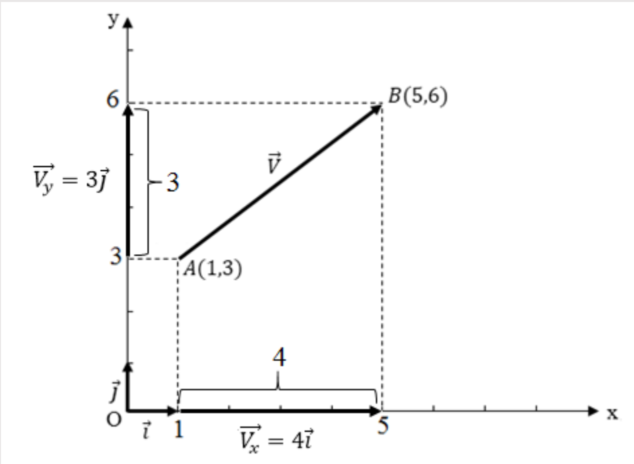
The vector $\vec{V}$ is written as: $\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} = (x_B - x_A)\vec{\imath} + (y_B - y_A)\vec{\jmath}$

Example

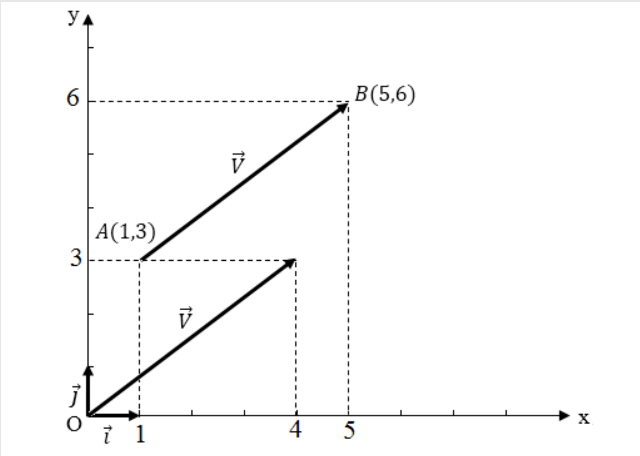
Consider a vector $\vec{V}$ *with starting point* $A(1,3)$ *and ending point* $B(5,6)$ *respectively.*

Scalar coordinates of the vector $\vec{V}$ *are:* $\begin{cases} V_x = x_B - x_A = 5 - 1 = 4 \\ V_y = y_B - y_A = 6 - 3 = 3 \end{cases}$

Vector components are: $\begin{cases} \vec{V_x} = 4\vec{\imath} \\ \vec{V_y} = 3\vec{\jmath} \end{cases} \Rightarrow \vec{V} = 4\vec{\imath} + 3\vec{\jmath}$



The obtained vector $\vec{V} = 4\vec{\imath} + 3\vec{\jmath}$ *have the scalar components* $\vec{V}\binom{4}{3}$, *we can draw this vector from the origin point (O) of the space with these components* $\binom{4}{3}$. *In this case we obtain a vector equal and parallel to the vector which is drawn from the beginning point A(1,3). We can say that we translate the vector to the origin point; there is no difference between these two vectors because they are parallel with the same magnitude.*



Application

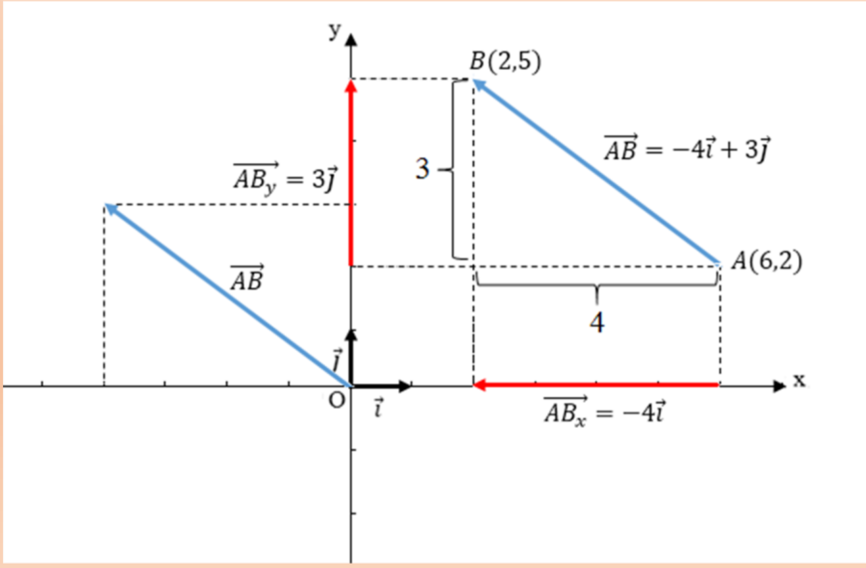
Consider two points $A(6,2)$ and $B(2,5)$. Found scalar components of the vector $\overrightarrow{AB}$ and draw it.

Solution:

The scalar components of the vector $\overrightarrow{AB}$ are: $\begin{cases} AB_x = x_B - x_A = 2 - 6 = -4 \\ AB_y = y_B - y_A = 5 - 2 = 3 \end{cases}$

The vector components are: $\begin{cases} \overrightarrow{AB_x} = -4\vec{\imath} \\ \overrightarrow{AB_y} = 3\vec{\jmath} \end{cases} \Rightarrow \vec{V} = -4\vec{\imath} + 3\vec{\jmath}$

By translation we can draw a vector from the origin point, equal to $\overrightarrow{AB}$ vector with components $\binom{-4}{3}$.



I.3.1.2 Magnitude of vector

The magnitude of a vector, is calculated from its scalar components V_x and V_y, is calculated as follows:

$$\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} \Rightarrow \boxed{\|\vec{V}\| = \sqrt{V_x^{\,2} + V_y^{\,2}}}$$

The vector $\vec{V}$ make an angle θ with positive x axis. If we apply the Pythagorean theorem on the right triangle we found: $\boldsymbol{V^2 = V_x{}^2 + V_y{}^2}$

Example

The magnitudes of vectors: $\vec{V_1} = 4\vec{\imath} - 2\vec{\jmath}$ *and* $\vec{V_2} = 3\vec{\imath} + \vec{\jmath}$ *are:*

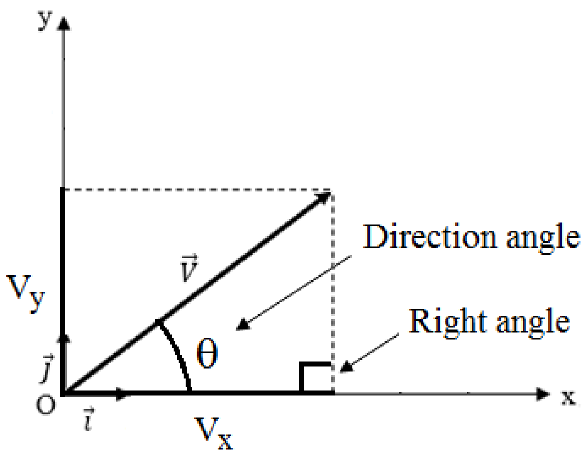
$\|\vec{V}_1\| = \sqrt{4^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$

$\|\vec{V}_2\| = \sqrt{3^2 + 1^2} = \sqrt{9 + 1} = \sqrt{10}$

I.3.1.3 Direction angle of vector

The direction angle is the angle that the vector forms with the positive direction on the x-axis (the angle is measured in the counterclockwise direction from the positive x-axis to the vector).

The direction angle of a vector is defined via the tangent function of the angle: $\boldsymbol{\tan\theta = \frac{V_y}{V_x}}$



We can calculate the components of a vector using the right triangle;

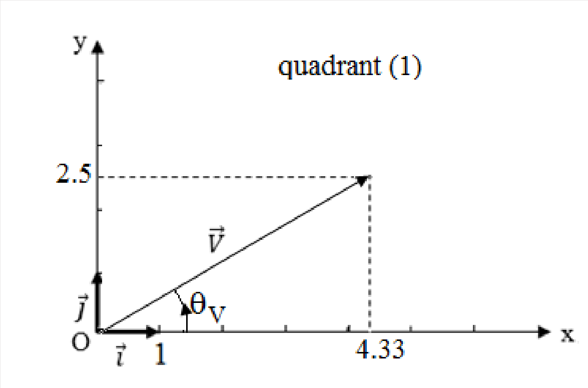
$$\begin{cases} \cos\theta = \frac{V_x}{\|\vec{V}\|} \\ \sin\theta = \frac{V_y}{\|\vec{V}\|} \end{cases} \Rightarrow \begin{cases} V_x = \|\vec{V}\|.\cos\theta \\ V_y = \|\vec{V}\|.\sin\theta \end{cases}$$

- ❖ If the vector lies in the first quadrant, the direction angle θ is positive and it is measured in the counterclockwise direction from the x-axis.

Example

For the vector in figure, the direction angle of the vector $\vec{V}$ is θ_V calculated as:

$$tan\,\theta_V = \frac{2.5}{4.33} = 0.577 \;\Rightarrow\; \theta_V = tan^{-1}\,0.577 = 29.98 \simeq 30°$$



❖ If the vector lies in the second quadrant, the direction angle θ_V is measured as:

$$\theta_V = \theta + 180°$$

Were θ is the angle between the negative x-axis and the vector, it is calculated as: tan $\theta = \frac{V_y}{V_x}$

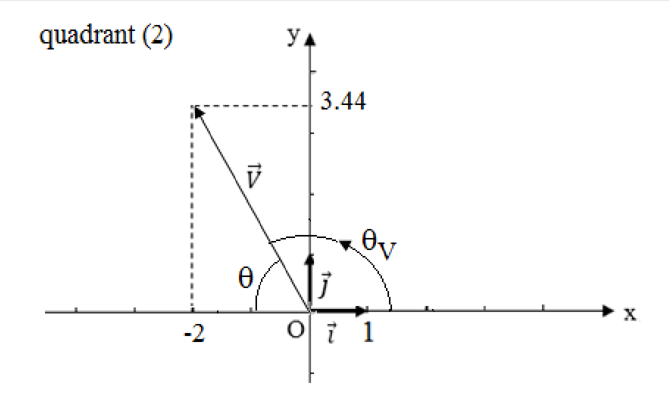
As Vx is negative, the angle θ is negative. We found: $\theta_V = \theta + 180°$

Example

For the vector in figure, the direction angle of the vector $\vec{V}$ is θ_V calculated as:

$$tan\,\theta = \frac{3.44}{-2} = -1.72 \;\Rightarrow\; \theta = tan^{-1}(-1.72) = -59.8° \simeq -60° \;\;;$$

$$\theta_V = \theta + 180° = -60° + 180° = 120°$$

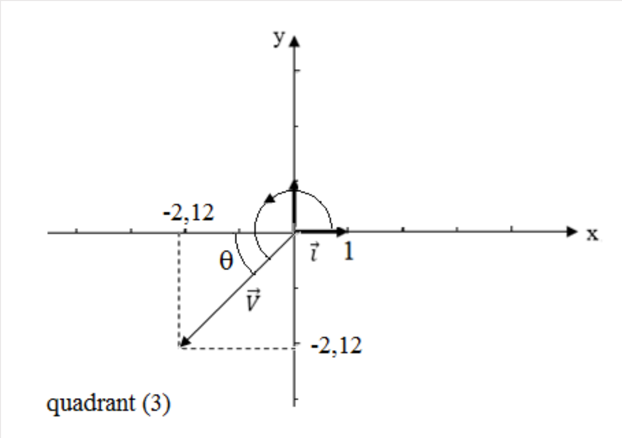


❖ If the vector lies in the third quadrant, the direction angle θ_V is measured like the case of the vector in the second quadrant: $\theta_V = \theta + 180°$

Example

For the vector in figure, the direction angle of the vector $\vec{V}$ is θ_V, calculated as:

$$tan\,\theta = \frac{-2.12}{-2.12} = +1 \Rightarrow \theta = tan^{-1}(+1) = +45° \;;\theta_V = \theta + 180° = 45 + 180° = 225°$$

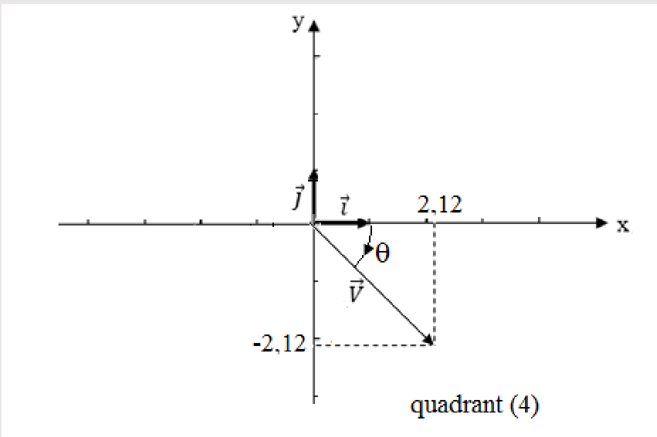


❖ If the vector lies in the fourth quadrant, the direction angle θ_V is measured clockwise from the positive x-axis direction; in this case, the direction angle is negative.

Example

For the vector in figure, the direction angle of the vector $\vec{V}$ is calculated as:

$$tan\,\theta = \frac{-2.12}{+2.12} = -1 \Rightarrow \theta = tan^{-1}(-1) = -45°$$



Application

Given two vectors $\overrightarrow{V_1} = -4\vec{\imath} + 2{,}9\vec{\jmath}$ *and* $\overrightarrow{V_2} = -3\vec{\imath} - 1{,}5\vec{\jmath}$ *; found the magnitude and the direction angle for each vector.*

Solution:

For vector $\overrightarrow{V_1}$: $\|\overrightarrow{V_1}\| = \sqrt{4^2 + (2{,}9)^2} = 4.94$

$V_x = -4$ *is negative, the vector lies in the second quadrant:* $\tan\theta = \frac{2{,}9}{-4} = -0.72$
$\Rightarrow \theta = -35{,}94°$

$\theta_{V1} = \theta + 180° = -35{,}94 + 180° = 144.06°$

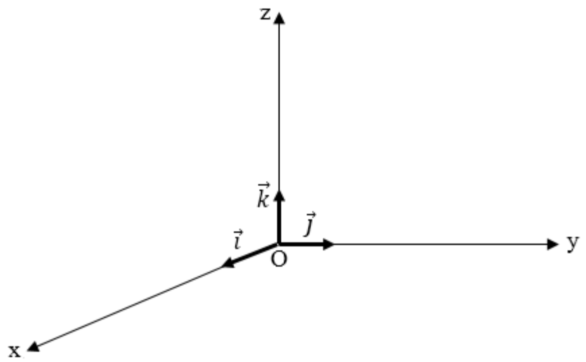
For vector $\overrightarrow{V_2}$: $\|\overrightarrow{V_2}\| = \sqrt{3^2 + (1{,}5)^2} = 3{,}35$

$V_x = -3$, $V_y = -1{,}5$, *the vector lies in the third quadrant:* $\tan\theta = \frac{-1{,}5}{-3} = 0{,}5 \quad \Rightarrow$
$\theta = 26{,}56°$

$\theta_{V2} = \theta + 180° = 26.56 + 180 = 206{,}56°$

I.3.2 Three-dimensional space

Three-dimensional space (characterized by an ordered triplet of axes (Oxyz) each of its axes is perpendicular to the others, the origin point O and the unit vectors $(\vec{\imath}, \vec{\jmath}, \vec{k})$ of a single unit of length $\|\vec{\imath}\| = \|\vec{\jmath}\| = \|\vec{k}\| = 1$).



A vector $\vec{V}$ in three-dimensional space is described by a triplet of its **vector components**:

$\vec{V} = \overrightarrow{V_x} + \overrightarrow{V_y} + \overrightarrow{V_z}$

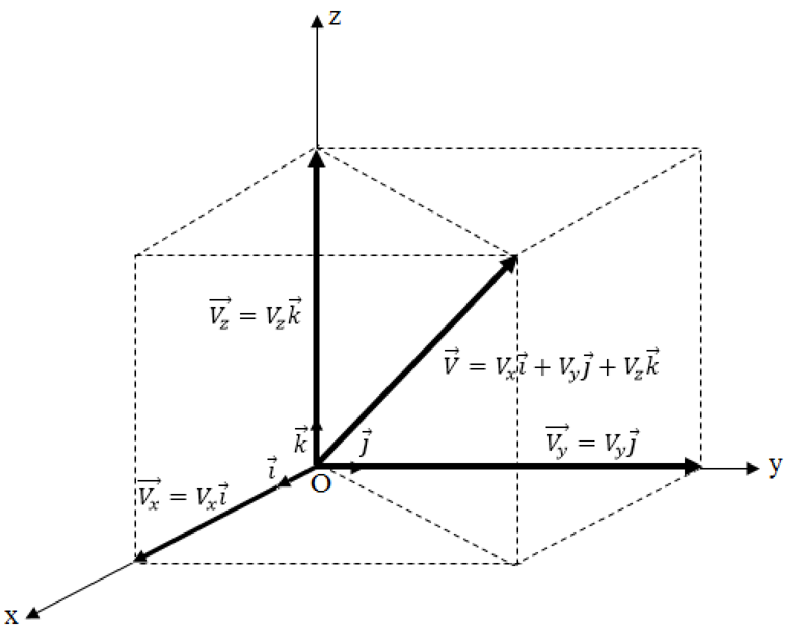
$\overrightarrow{V_x}$ the x-coordinate called the x component, $\overrightarrow{V_y}$ the y-coordinate called the y component and $\overrightarrow{V_z}$ the z-coordinate called the z component.

We can write: $\overrightarrow{V_x} = V_x\vec{\imath}$; $\overrightarrow{V_y} = V_y\vec{\jmath}$; $\overrightarrow{V_z} = V_z\vec{k}$

$V_x = \|\overrightarrow{V_x}\|$, $V_y = \|\overrightarrow{V_y}\|$ and $V_z = \|\overrightarrow{V_z}\|$ are the scalar components of the vector $\vec{V}$ and we write:

$$\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k} \iff \vec{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

The magnitude of the vector $\vec{V}$ is: $\boxed{\|\vec{\boldsymbol{V}}\| = \sqrt{\boldsymbol{V_x}^2 + \boldsymbol{V_y}^2 + \boldsymbol{V_z}^2}}$

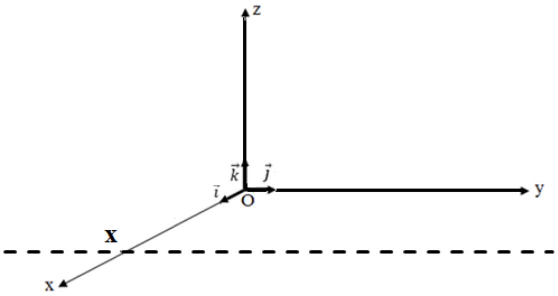


Plotting points and vectors in space

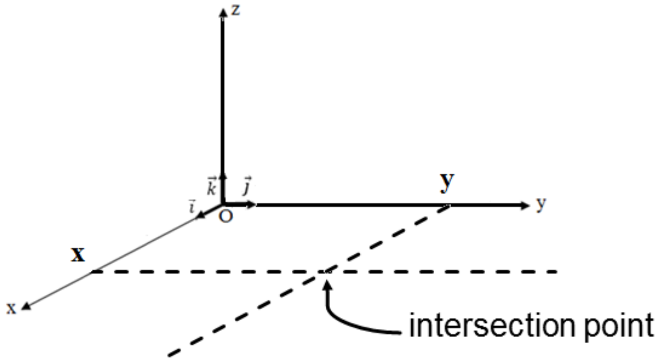
To Plot a point M of coordinates (x, y, z) on in three-dimensional space;

We follow the following steps:

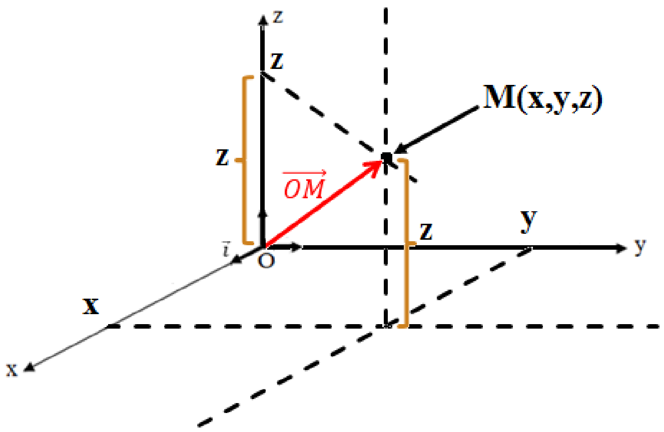
First, we locate the coordinate x on x-axis, then from this point, draw a dashed line pallel to y-axis



Second, locate the coordinate y on y-axis, then from this point; draw a dashed line parallel to x-axis.



Third, from the intersection point we draw a dashed line parallel to z-axis.



Finally, we locate the coordinate z on z-axis, then from this point; draw a dashed line, parallel to Oxy plane, with the same level with z length.

The intersection point represents the M point.

To draw the vector $\overrightarrow{OM}$, we draw an arrow from the origin O to the M point.

I.3.3 Types of vectors

I.3.3.1 Unit vectors

A unit vector is a vector that has a magnitude equal to 1. It describes a direction in the space.

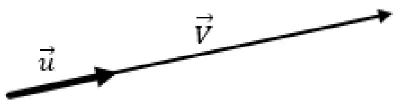
Each space (two-dimensional or three-dimensional) were always attached by the unit vectors $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$, where $\|\vec{\imath}\| = \|\vec{\jmath}\| = \|\vec{k}\| = 1$

For vectors, generally the unit vector is denoted by $\vec{u}$. Each vector has a unit vector describe the direction of the vector. The vector and the unit vector are parallel.

The vector $\vec{V}$ has the unit vector $\vec{u}$, where $\vec{V} = \|\vec{V}\|.\vec{u} = V.\vec{u}$

So: $\vec{u} = \frac{\vec{V}}{\|\vec{V}\|}$

The magnitude of the unit vector $\vec{u}$ is always equal to 1.



Example

The unit vector of the vector $\overrightarrow{V_1} = 2\sqrt{2}\vec{\imath} + \vec{\jmath} + 3\vec{k}$ *is:*

$$\vec{u} = \frac{\overrightarrow{V_1}}{\|\overrightarrow{V_1}\|} = \frac{2\sqrt{2}\vec{\imath} + \vec{\jmath} + 3\vec{k}}{\sqrt{(2\sqrt{2})^2 + 1^2 + 3^2}} = \frac{2\sqrt{2}\vec{\imath} + \vec{\jmath} + 3\vec{k}}{\sqrt{18}}$$

So: $\vec{u} = \frac{2\sqrt{2}}{\sqrt{18}}\vec{\imath} + \frac{1}{\sqrt{18}}\vec{\jmath} + \frac{3}{\sqrt{18}}\vec{k}$

The magnitude of $\vec{u}$ *is:* $\vec{u} = \sqrt{\left(\frac{2\sqrt{2}}{\sqrt{18}}\right)^2 + \left(\frac{1}{\sqrt{18}}\right)^2 + \left(\frac{3}{\sqrt{18}}\right)^2} = \sqrt{\frac{8}{18} + \frac{1}{18} + \frac{9}{18}} = \sqrt{\frac{18}{18}} = 1$

I.3.3.2 Zero Vector

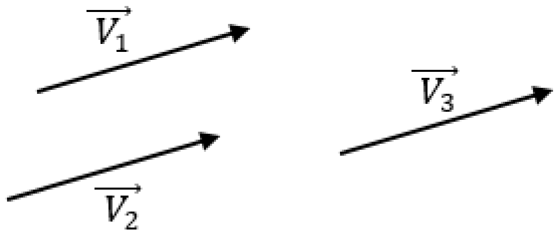
If the vector $\overrightarrow{AB} = \vec{0}$ (equal to zero vector); that is means that the two points A and B are superimposed and the magnitude of the vector is zero.

In a Cartesian basis, a zero vector is given by: $\vec{0} = 0\vec{\imath} + 0\vec{\jmath} + 0\vec{k}$

I.3.3.3 Free vector

A free vector is a vector defined by its direction and module without fixing its point of application.

The vectors $\overrightarrow{V_1}$, $\overrightarrow{V_2}$ and $\overrightarrow{V_3}$ represent the same vector $\vec{V}$. They have the same direction, same magnitude and variable application point.



I.3.3.4 Coplanar vectors

The vectors lying to the same plane or parallel to the same plane are called coplanar vectors.

I.3.3.5 Collinear vectors

The vectors lying to either the same line or parallel lines are called collinear vectors; two collinear vectors are parallel.

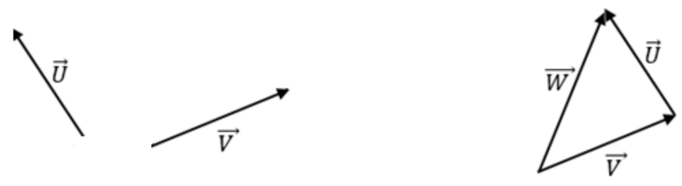
I.3.3.6 Negative of a vector

If two vectors have the same magnitude with an opposite direction, the two vectors have opposite signs. We write $\overrightarrow{AB} = -\overrightarrow{BA}$

I.3.4 Adding and subtracting vectors

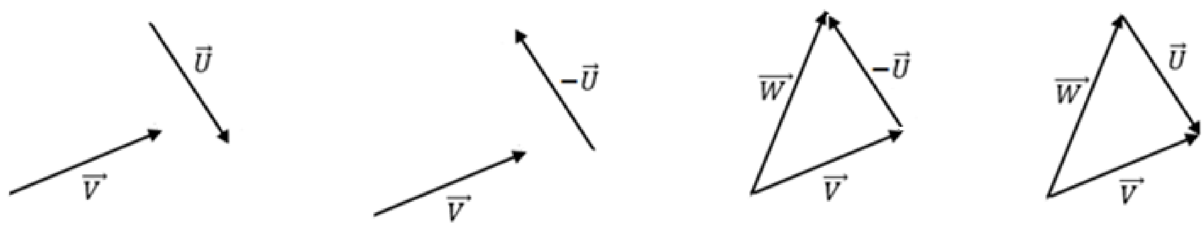
Two vectors can be added by placing the head of the first vector with the tail of the second one after translation of one of both vectors. The resultant vector is draws from the tail of the first vector to the head of the second (figure).

The sum of the two vectors $\overrightarrow{V}$ and $\vec{U}$ is $\overrightarrow{W} = \vec{V} + \vec{U}$



The subtraction of two vectors $\overrightarrow{V}$ and $\vec{U}$ is a vector $\overrightarrow{W} = \vec{V} - \vec{U}$

We can note that $\overrightarrow{W} = \vec{V} + (-\vec{U})$; $\overrightarrow{W}$ is the addition of $\vec{V}$ and $(-\vec{U})$, so we can obtain $\overrightarrow{W}$ by placing the head of $\vec{V}$ vector with the tail of $(-\vec{U})$ vector



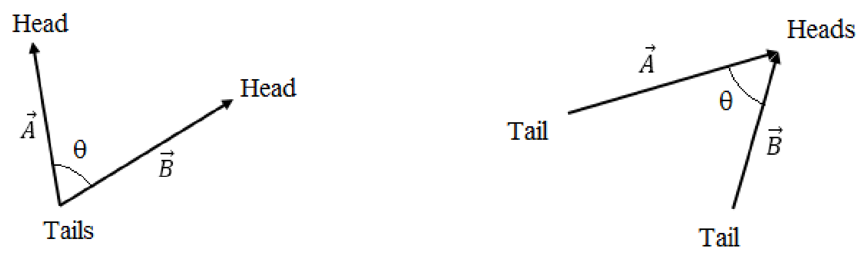
The sum and subtraction of two vectors in Cartesian system can performed by adding and subtracting the components.

$$\overrightarrow{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{K} \ ; \quad \vec{U} = U_x\vec{\imath} + U_y\vec{\jmath} + U_z\vec{K}$$

$$\vec{V} - \vec{U} = (V_x - U_x)\vec{\imath} + \left(V_y - U_y\right)\vec{\jmath} + (V_z - U_z)\vec{K}$$

I.3.4.1 Angle between two vectors

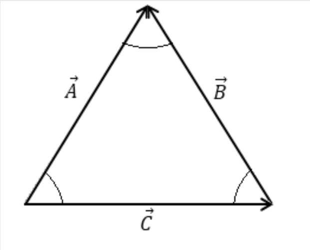
The angle between two vectors when the tails or heads of both the vectors coincide lies between $0° \leq \theta \leq 180°$. Otherwise, i.e. if the head of a vector coincide with the tail of the second vector, the angle between vectors is calculated. In this case we should translate a vector in order to coincide the tails or the heads (often coincide the tails) and then calculate the angle.



Example

Find the angle between vectors (If they form an equilateral triangle).

a- $\vec{A}$ *and* $\vec{B}$ *vector*
b- $\vec{B}$ *and* $\vec{C}$ *vector*
c- $\vec{A}$ *and* $\vec{C}$ *vector*

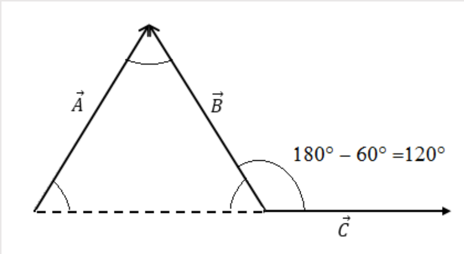


Solution:

a- *For vectors* $\vec{A}$ *and* $\vec{B}$*: head of both vectors coincide with each other, hence the angle of an equilateral triangle is equal to 60°, so the angle between* $\vec{A}$ *and* $\vec{B}$ *vector is 60°.*

b- *For vectors* $\vec{B}$ *and* $\vec{C}$*: head or tail of the* $\vec{B}$ *and* $\vec{C}$ *vector does not coincide with each other so we need to transmit a vector. We translate the vector* $\vec{C}$ *until the tails of both vectors became coincide and we calculate the angle:* $180 - 60 = 120°$

So, the angle between $\vec{B}$ *and* $\vec{C}$ *is 120°.*



c- *For vectors* $\vec{A}$ *and* $\vec{C}$*: tails of both vectors coincide with each other; the vector between the two vectors is 60°.*

I.3.4.2 Properties of vector addition and subtraction

Addition of vectors

Commutative: $\overrightarrow{V_1} + \overrightarrow{V_2} = \overrightarrow{V_2} + \overrightarrow{V_1}$

Associativity: $\overrightarrow{V_1} + (\overrightarrow{V_2} + \overrightarrow{V_3}) = (\overrightarrow{V_1} + \overrightarrow{V_2}) + \overrightarrow{V_3}$

The zero vector ($\vec{0}$) is a neutral element; $\vec{V} + \vec{0} = \vec{V}$.

Distributive with a number: $\alpha(\overrightarrow{V_1} + \overrightarrow{V_2}) = \alpha\overrightarrow{V_1} + \alpha\overrightarrow{V_2}$

$$(\alpha + \beta)\overrightarrow{V_1} = \alpha\overrightarrow{V_1} + \beta\overrightarrow{V_1}$$

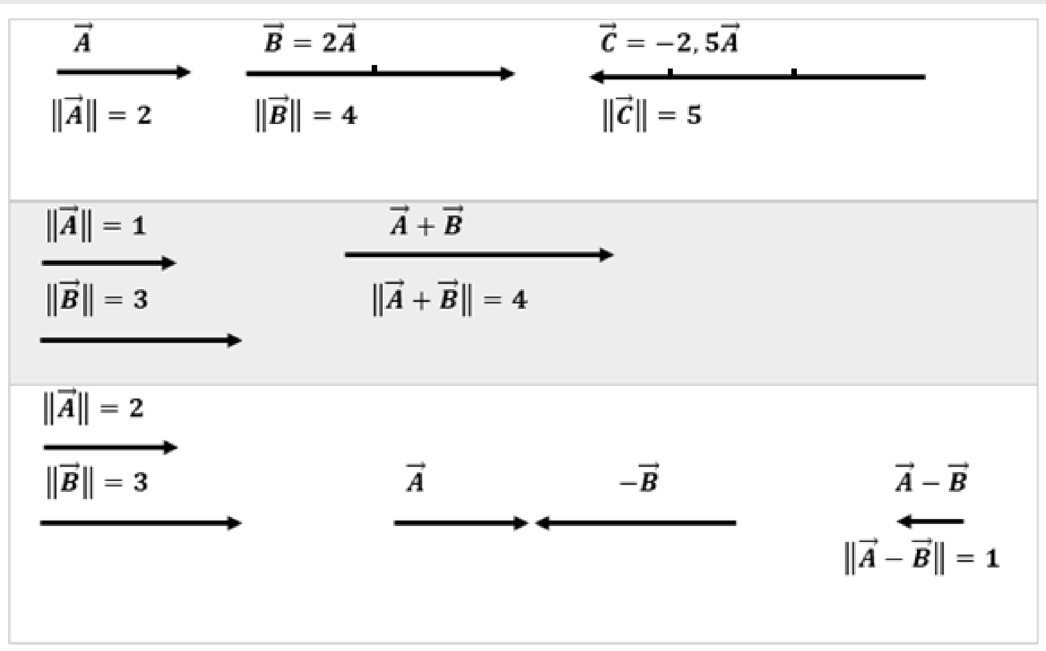
Subtraction of vectors

Not commutative: $\overrightarrow{V_1} - \overrightarrow{V_2} \neq \overrightarrow{V_2} + \overrightarrow{V_1}$

Not associative: $\overrightarrow{V_1} - (\overrightarrow{V_2} - \overrightarrow{V_3}) \neq (\overrightarrow{V_1} - \overrightarrow{V_2}) - \overrightarrow{V_3}$

Any vector subtracted from itself results in a zero vector: $\vec{V} - \vec{V} = \vec{0}$.

Example



Application 1

Given three vectors $\vec{V_1}$, $\vec{V_2}$ and $\vec{V_3}$:

$\vec{V_1} = 2\vec{\imath} - \vec{\jmath} + 2\vec{k}$; $\vec{V_2} = \vec{\imath} + 2\vec{\jmath} + 3\vec{k}$ *and* $\vec{V_3} = 2\vec{\imath} + 2\vec{\jmath} + 5\vec{k}$

- *Calculate $\vec{V_4}$ the sum of the three vectors.*
- *Calculate the modules of the three vectors and the resulting vector.*
- *Represent these vectors.*

Solution:

$$\vec{V_4} = \vec{V_1} + \vec{V_2} + \vec{V_3} = (2-1+2)\vec{\imath} + (1+2+3)\vec{\jmath} + (2+2+5)\vec{k}$$

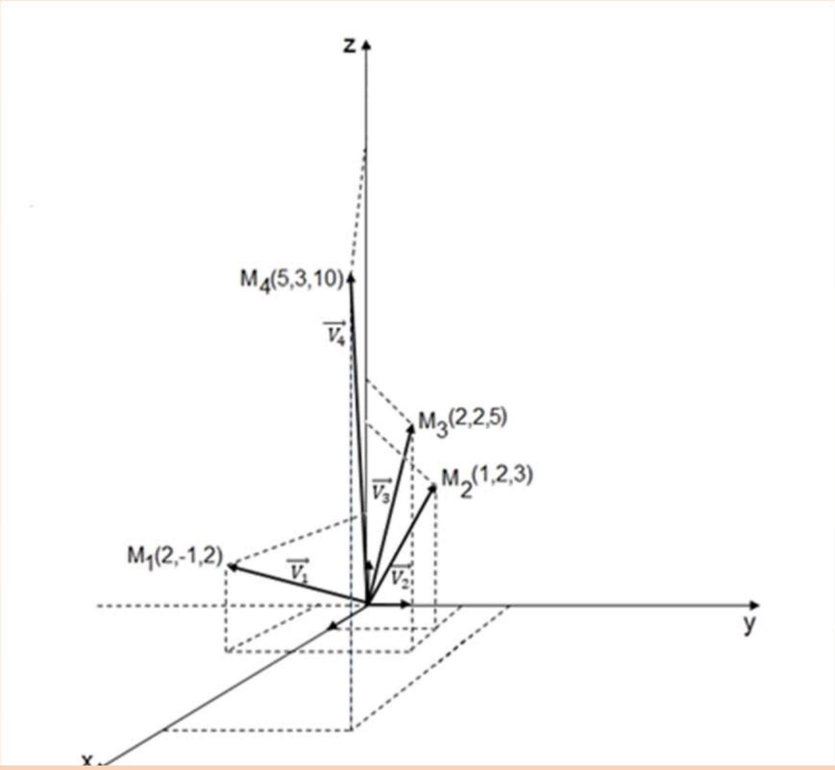
So, $\vec{V_4} = 5\vec{\imath} + 3\vec{\jmath} + 10\vec{k}$

$$\|\vec{V_1}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

$$\|\vec{V_2}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\|\vec{V_3}\| = \sqrt{2^2 + 2^2 + 5^2} = \sqrt{33}$$

$$\|\vec{V_4}\| = \sqrt{5^2 + 3^2 + 10^2} = \sqrt{134}$$



I.3.5 Multiplication of vectors

There are two kinds of multiplication of vectors: scalar product and vector product. Scalar product gives a scalar quantity although vector product gives a vector quantity.

I.3.5.1 Scalar product

The scalar product of two vectors $\overrightarrow{V_1}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\overrightarrow{V_2}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ has two expressions:

Analytical expression of scalar product

The analytical expression consists to use the components of vectors:

$$\overrightarrow{V_1}.\overrightarrow{V_2} = (x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}).(x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k})$$

$$= x_1x_2\vec{\imath}.\vec{\imath} + x_1y_2\vec{\imath}.\vec{\jmath} + x_1z_2\vec{\imath}.\vec{k} + y_1x_2\vec{\jmath}.\vec{\imath} + y_1y_2\vec{\jmath}.\vec{\jmath} + y_1z_2\vec{\jmath}.\vec{k} + z_1x_2\vec{k}.\vec{\imath} + z_1y_2\vec{k}.\vec{\jmath} + z_1z_2\vec{k}.\vec{k}$$

$$\boxed{\boldsymbol{\overrightarrow{V_1}.\overrightarrow{V_2} = x_1x_2 + y_1y_2 + z_1z_2}}$$

Because the unit $\vec{\imath}, \vec{\jmath}\ et\ \vec{k}$ vectors are orthogonal so:

$$\boldsymbol{\vec{\imath}.\vec{\jmath} = \|\vec{\imath}\|.\|\vec{\jmath}\|.\cos 90° = (1).(1).(0) = 0}$$

$$\boldsymbol{\vec{\imath}.\vec{k} = \|\vec{\imath}\|.\|\vec{k}\|.\cos 90° = (1).(1).(0) = 0}$$

$$\boldsymbol{\vec{\jmath}.\vec{k} = \|\vec{\imath}\|.\|\vec{k}\|.\cos 90° = (1).(1).(0) = 0}$$

And:

$$\vec{\imath}.\vec{\imath} = \|\vec{\imath}\|.\|\vec{\imath}\|.\cos 0° = (\mathbf{1}).(\mathbf{1}).(\mathbf{1}) = 1$$

$$\vec{\jmath}.\vec{\jmath} = \|\vec{\jmath}\|.\|\vec{\jmath}\|.\cos 0° = (\mathbf{1}).(\mathbf{1}).(\mathbf{1}) = 1$$

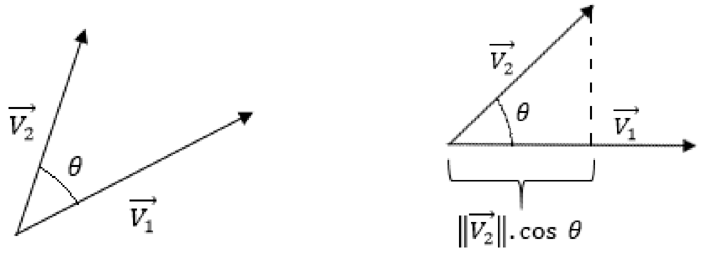
$$\vec{k}.\vec{k} = \|\vec{k}\|.\|\vec{k}\|.\cos 0° = (\mathbf{1}).(\mathbf{1}).(\mathbf{1}) = 1$$

Geometric expression of scalar product

The result of the scalar product of two vectors is a scalar. The geometric formula of the scalar product is:

$$\boxed{\boldsymbol{\overrightarrow{V_1}.\overrightarrow{V_2} = \|\overrightarrow{V_1}\|.\|\overrightarrow{V_2}\|.\cos\theta}}$$

θ is the angle between the two vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$.



Scalar multiplication of vectors is commutative: $\overrightarrow{V_1}.\overrightarrow{V_2} = \overrightarrow{V_2}.\overrightarrow{V_1}$

It is also distributive: $\overrightarrow{V_1}.\left(\overrightarrow{V_2} + \overrightarrow{V_3}\right) = \overrightarrow{V_1}.\overrightarrow{V_2} + \overrightarrow{V_1}.\overrightarrow{V_3}$

We can calculate the angle θ between the two vectors using geometric expression of the scalar product: $\boldsymbol{\cos\theta} = \frac{\overrightarrow{\boldsymbol{V_1}}.\overrightarrow{\boldsymbol{V_2}}}{\|\overrightarrow{\boldsymbol{V_1}}\|.\|\overrightarrow{\boldsymbol{V_2}}\|}$

We use the analytical expression of scalar product to calculate the product $\overrightarrow{V_1}.\overrightarrow{V_2}$:

$$\boldsymbol{\theta = \cos^{-1}\left(\frac{x_1x_2 + y_1y_2 + z_1z_2}{\|\overrightarrow{V_1}\|.\|\overrightarrow{V_2}\|}\right)}$$

Example:

1/ The angle between the vectors $\overrightarrow{V_1} = 5\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$ and $\overrightarrow{V_2} = 6\vec{\imath} + 6\vec{\jmath} + \vec{k}$ is θ_1:

$\|\overrightarrow{V_1}\| = \sqrt{33}$; $\|\overrightarrow{V_2}\| = \sqrt{73}$

$$\theta_1 = \cos^{-1}\left(\frac{5.6 - 2.6 + 2.2}{\sqrt{33}.\sqrt{73}}\right) = \cos^{-1}(0.4) = 66{,}42°$$

2/ The angle between the vectors $\overrightarrow{V_1} = 5\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$ and $\overrightarrow{V_3} = -6\vec{\imath} + 6\vec{\jmath} + \vec{k}$ is θ_2:

$$\theta_2 = \cos^{-1}\left(\frac{-6.5 - 2.6 + 2.1}{\sqrt{33}.\sqrt{73}}\right) = \cos^{-1}(-0.81) = 144.09°$$

I.3.5.2 vector product

The result of the vector product (named also cross product) of two vectors is a vector. The vector product of two vectors $\overrightarrow{V_1}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\overrightarrow{V_2}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ is denoted: $\boldsymbol{\overrightarrow{V_1} \wedge \overrightarrow{V_2}}$. This product has two expressions:

Geometric expression of vector product

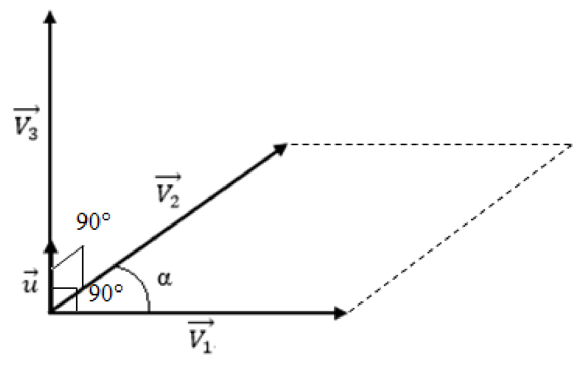
The geometric expression of the vector product is as follows:

$$\boxed{\boldsymbol{\overrightarrow{V_1} \wedge \overrightarrow{V_2} = \|\overrightarrow{V_1}\|.\|\overrightarrow{V_2}\|.\sin\alpha\,.\vec{u}}}$$

α is the angle between the two vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ (from the first vector $\overrightarrow{V_1}$ to the second $\overrightarrow{V_2}$), $\vec{u}$ is the unit vector of the resultant vector.

The vector product $\overrightarrow{V_1} \wedge \overrightarrow{V_2}$ is a vector that has its direction perpendicular to both vectors. In other words, the vector $\overrightarrow{V_1} \wedge \overrightarrow{V_2}$ is perpendicular to the plane that contains vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$.

- If two vectors are parallel, there vector product must be zero: $\boldsymbol{\overrightarrow{V_1} // \overrightarrow{V_2} \Leftrightarrow \overrightarrow{V_1} \wedge \overrightarrow{V_2} = \vec{0}}$
- If $\overrightarrow{V_3}$ is the resultant vector, we write: $\overrightarrow{V_3} = \overrightarrow{V_1} \wedge \overrightarrow{V_2}$



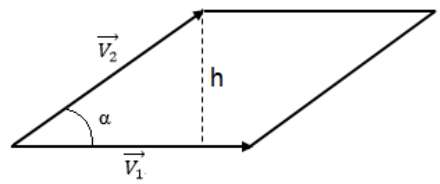
- The magnitude of the vector product is defined as:

$$\boldsymbol{\|\overrightarrow{V_1} \wedge \overrightarrow{V_2}\| = \|\overrightarrow{V_1}\|.\|\overrightarrow{V_2}\|.\sin\alpha}$$

We note that $(V_2.\sin\alpha)$ represents the width of the parallelogram formed by the two vectors.

$$\sin\alpha = \frac{h}{\|\overrightarrow{V_2}\|} \Rightarrow h = \|\overrightarrow{V_2}\|.\sin\alpha$$

So: $\left\|\overrightarrow{\boldsymbol{V_1}} \wedge \overrightarrow{\boldsymbol{V_2}}\right\| = \left\|\overrightarrow{\boldsymbol{V_1}}\right\|.\boldsymbol{h}$



We conclude that $\left\|\overrightarrow{V_1} \wedge \overrightarrow{V_2}\right\| = \left\|\overrightarrow{V_1}\right\|.h$ represents the area of the parallelogram formed by the two vectors.

Analytical expression of vector product

The vector product of the two vectors is expressed analytically by:

$$\overrightarrow{V_1}\wedge\overrightarrow{V_2} = \left(x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}\right)\wedge\left(x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}\right)$$

$$= x_1\vec{\imath}\wedge\left(x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}\right) + y_1\vec{\jmath}\wedge\left(x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}\right) + z_1\vec{k}\wedge\left(x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}\right)$$

$$= x_1x_2\vec{\imath}\wedge\vec{\imath} + x_1y_2\vec{\imath}\wedge\vec{\jmath} + x_1z_2\vec{\imath}\wedge\vec{k} + y_1x_2\vec{\jmath}\wedge\vec{\imath} + y_1y_2\vec{\jmath}\wedge\vec{\jmath} + y_1z_2\vec{\jmath}\wedge\vec{k} + z_1x_2\vec{k}\wedge\vec{\imath} + z_1y_2\vec{k}\wedge\vec{\jmath} + z_1z_2\vec{k}\wedge\vec{k}$$

We have: $\vec{\imath}\wedge\vec{\jmath} = \vec{k}$; $\vec{\jmath}\wedge\vec{k} = \vec{\imath}$;$\vec{k}\wedge\vec{\imath} = \vec{\jmath}$ et $\vec{\imath}\wedge\vec{\imath} = \vec{\jmath}\wedge\vec{\jmath} = \vec{k}\wedge\vec{k} = \vec{0}$

$$\overrightarrow{V_1}\wedge\overrightarrow{V_2} = x_1y_2\vec{k} - x_1z_2\vec{\jmath} - y_1x_2\vec{k} + y_1z_2\vec{\imath} + z_1x_2\vec{\jmath} - z_1y_2\vec{\imath}$$

$$\boxed{\overrightarrow{\boldsymbol{V_1}}\wedge\overrightarrow{\boldsymbol{V_2}} = (\boldsymbol{y_1z_2 - z_1y_2})\vec{\boldsymbol{\imath}} - (\boldsymbol{x_1z_2 - z_1x_2})\vec{\boldsymbol{\jmath}} + (\boldsymbol{x_1y_2 - y_1x_2})\vec{\boldsymbol{k}}}$$

e can find this expression easily using the determinant form (of order 3):

$$\overrightarrow{\boldsymbol{V_1}}\wedge\overrightarrow{\boldsymbol{V_2}} = \begin{vmatrix} \vec{\boldsymbol{\imath}} & \vec{\boldsymbol{\jmath}} & \vec{\boldsymbol{k}} \\ \boldsymbol{x_1} & \boldsymbol{y_1} & \boldsymbol{z_1} \\ \boldsymbol{x_2} & \boldsymbol{y_2} & \boldsymbol{z_2} \end{vmatrix} = \begin{vmatrix} \boldsymbol{y_1} & \boldsymbol{z_1} \\ \boldsymbol{y_2} & \boldsymbol{z_2} \end{vmatrix}\vec{\boldsymbol{\imath}} - \begin{vmatrix} \boldsymbol{x_1} & \boldsymbol{z_1} \\ \boldsymbol{x_2} & \boldsymbol{z_2} \end{vmatrix}\vec{\boldsymbol{\jmath}} + \begin{vmatrix} \boldsymbol{x_1} & \boldsymbol{y_1} \\ \boldsymbol{x_2} & \boldsymbol{y_2} \end{vmatrix}\vec{\boldsymbol{k}}$$

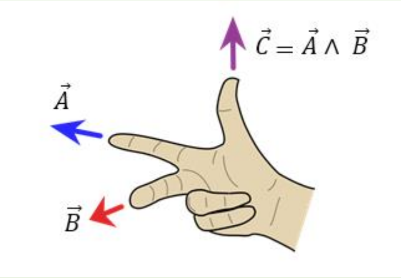
We simplify the determinant to 2x2 order and then execute the product as following:

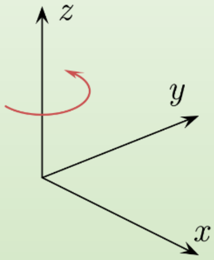
$$\begin{vmatrix} \boldsymbol{y_1} & \boldsymbol{z_1} \\ \boldsymbol{y_2} & \boldsymbol{z_2} \end{vmatrix} = \left(y_1z_2 - z_1y_2\right)$$

$$\boxed{\overrightarrow{\boldsymbol{V_1}}\wedge\overrightarrow{\boldsymbol{V_2}} = (\boldsymbol{y_1z_2 - z_1y_2})\vec{\boldsymbol{\imath}} - (\boldsymbol{x_1z_2 - z_1x_2})\vec{\boldsymbol{\jmath}} + (\boldsymbol{x_1y_2 - y_1x_2})\vec{\boldsymbol{k}}}$$

Note:

The **right-hand screw rule** is a method used to determine the direction of the resultant vector of vector product. The right-hand screw rule requires taking three fingers (thumb, index finger and middle finger); the product of two successive fingers gives the third finger in manner choose the counterclockwise as direction of rotation.

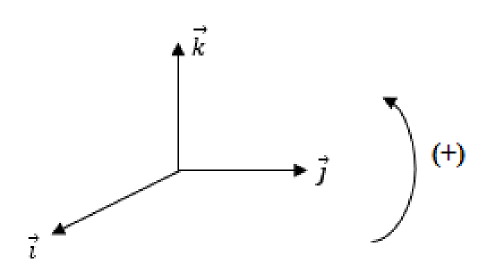




Take the index finger along the first vector ($\vec{A}$), then the middle finger along the second vector ($\vec{B}$), then the thumb along the third vector ($\vec{C}$). The vector product of $\vec{A} \wedge \vec{B}$ equal $\vec{C}$.

Take the example of the three unit vectors $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$;

$$\vec{\imath} \wedge \vec{\jmath} = \vec{k}\,;\, \vec{\jmath} \wedge \vec{k} = \vec{\imath}\,;\, \vec{k} \wedge \vec{\imath} = \vec{\jmath}$$



Properties of vector product

The vector product is anticommutative: $\vec{A} \wedge \vec{B} = -\vec{B} \wedge \vec{A}$

- Distributive: $\vec{A} \wedge (\vec{B} + \vec{C}) = \vec{A} \wedge \vec{B} + \vec{A} \wedge \vec{C}$

- Associative with multiplication by a real number λ: $\lambda(\vec{A} \wedge \vec{B}) = (\lambda\vec{A}) \wedge \vec{B} = \vec{A} \wedge (\lambda\vec{B})$

- Non-associative: $(\vec{A} \wedge \vec{B}) \wedge \vec{C} \neq \vec{A} \wedge (\vec{B} \wedge \vec{C})$

Example

Vectors $\vec{A}$ and $\vec{B}$ are given as $\vec{A} = 3\vec{i} + 4\vec{j} - 2\vec{k}$ and $\vec{B} = 2\vec{i} - 6\vec{j} + 3\vec{k}$.

1/ The vector product $\vec{A} \wedge \vec{B}$ is:

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & -2 \\ 2 & -6 & 3 \end{vmatrix} = \begin{vmatrix} 4 & -2 \\ -6 & 3 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & -2 \\ 2 & 3 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 4 \\ 2 & -6 \end{vmatrix}\vec{k}$$

$$= (4.3 - ((-6).(-2))\vec{i} - (3.3 - (2.(-2))\vec{j} + \left((3.(-6)) - 2.4\right)\vec{k}$$

$$\boxed{\vec{A} \wedge \vec{B} = -13\vec{i} - 26\vec{k}}$$

2/ The area (S) of the parallelogram formed by the two vectors $\vec{A}$ and $\vec{B}$ is:

$$S = \|\vec{A} \wedge \vec{B}\| = \sqrt{(13)^2 + (26)^2} = 29.06$$

I.3.6 Mixed product

The mixed product of three vectors is a product that contains both scalar and vector products between these vectors. The result of the mixed product is a scalar.

The analytical expression of the mixed product of three vectors $\vec{V_1}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \vec{V_2}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ and $\vec{V_3}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$ is as follows:

$$\vec{V_1}.(\vec{V_2} \wedge \vec{V_3}) = \vec{V_1}.\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \vec{V_1}.\left(\vec{i}.\begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - \vec{j}.\begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} + \vec{k}.\begin{vmatrix} x_2 & y_2 \\ y_3 & z_3 \end{vmatrix}\right)$$

$$\vec{V_1}.(\vec{V_2} \wedge \vec{V_3}) = x_1(y_2 z_3 - y_3 z_2) - y_1(x_2 z_3 - x_3 z_2) + z_1(x_2 y_3 - x_3 y_2) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

The geometric expression of the mixed product is:

$$\boxed{\vec{V_1}.(\vec{V_2} \wedge \vec{V_3}) = \|\vec{V_1}\|.\|\vec{V_2} \wedge \vec{V_3}\| \cos(\vec{V_1}, (\vec{V_2} \wedge \vec{V_3}))}$$

We put α the angle between $\vec{V_1}$ and $(\vec{V_2} \wedge \vec{V_3})$:

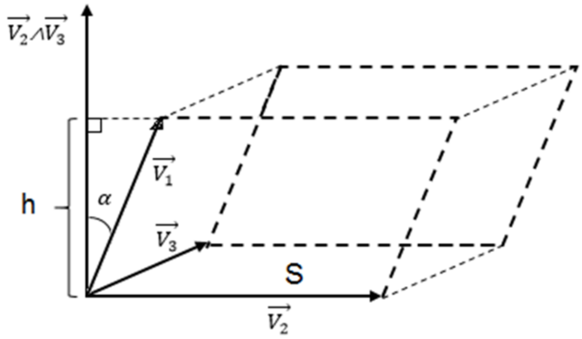
It is noticed that $(h = \|\vec{V_1}\|.\cos\alpha)$ represents the projection of vector $\vec{V_1}$ on the vector resulting from the vector product $\vec{V_2} \wedge \vec{V_3}$.

$\overrightarrow{V_1}.\left(\overrightarrow{V_2} \wedge \overrightarrow{V_3}\right) = \|\overrightarrow{V_1}\|.\|\overrightarrow{V_2} \wedge \overrightarrow{V_3}\| \cos\alpha = h.\|\overrightarrow{V_2} \wedge \overrightarrow{V_3}\|$

$\|\overrightarrow{V_2} \wedge \overrightarrow{V_3}\|$ represents the area (S) of the parallelogram constructed by the vectors $\overrightarrow{V_2}$ *and* $\overrightarrow{V_3}$. Therefore, we obtain the product of the surface via the width, which represents the volume (V) of a parallelepiped with edges formed by the three vectors.

$$V = \left|\overrightarrow{V_1}.\left(\overrightarrow{V_2} \wedge \overrightarrow{V_3}\right)\right| = h.S$$

Consequently, we can calculate the volume of a parallelepiped with edges formed by three vectors by using the mixed product of the three vectors (taking the absolute value of the product).



Example

The volume (V) of a parallelepiped with edges formed by the three vectors: $\vec{A} = 2\vec{\imath} - \vec{\jmath} + 3\vec{k}$

$\vec{B} = 3\vec{\imath} + 2\vec{\jmath} - 4\vec{k}$ *and* $\vec{C} = -2\vec{\imath} + \vec{k}$ *is:*

$V = \left|\vec{A}.\left(\vec{B} \wedge \vec{C}\right)\right|$

$$\vec{B} \wedge \vec{C} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & 2 & -4 \\ -2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ 0 & 1 \end{vmatrix}\vec{\imath} - \begin{vmatrix} 3 & -4 \\ -2 & 1 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} 3 & 2 \\ -2 & 0 \end{vmatrix}\vec{k} = 2\vec{\imath} + 5\vec{\jmath} + 4\vec{k}$$

$\vec{A}.\left(\vec{B} \wedge \vec{C}\right) = \left(2\vec{\imath} - \vec{\jmath} + 3\vec{k}\right).\left(2\vec{\imath} + 5\vec{\jmath} + 4\vec{k}\right) = 11$

The volume (V) of a parallelepiped is 11.

Properties:

- Circular permutation of vectors; which is a property of the determinant:

$$\vec{A}.(\vec{B} \wedge \vec{C}) = \vec{C}.(\vec{A} \wedge \vec{B}) = \vec{B}.(\vec{C} \wedge \vec{A})$$

- The mixed product is zero if the three vectors belong to the same plane (coplanar).

I.3.7 Derivative of vector product

Consider a scalar function λ(t) depending on time t and two vectors $\vec{A}$ and $\vec{B}$ also depending on time.

- $\frac{d(\vec{A}+\vec{B})}{dt} = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$
- $\frac{d\lambda\vec{A}}{dt} = \lambda.\frac{d\vec{A}}{dt} + \frac{d\lambda}{dt}.\vec{A}$
- $\frac{d(\vec{A}.\vec{B})}{dt} = \vec{A}.\frac{d\vec{B}}{dt} + \overrightarrow{B}.\frac{d\vec{A}}{dt}$
- $\frac{d(\vec{A}\wedge\vec{B})}{dt} = \frac{d\vec{A}}{dt} \wedge \vec{B} + \vec{A} \wedge \frac{d\vec{B}}{dt}$

I.4 Coordinate system

The position of a moving body is determined at each instant by coordinates associated with a reference. However, there are several types of coordinates and the choice of these depends on the mechanical problem or the type of body movement (rectilinear, curvilinear, circular, etc.).

I.4.1 Cartesian coordinates

The Cartesian coordinate allow to determine the position of a point on a line, a plane (two dimensions) or a space of three dimensions.

A point ***M*** in the Cartesian coordinate system (three-dimensional space) is characterized by an ordered triple coordinate (x, y, z): x is the **abscissa**, y is the **ordinate** and z is the **applicate** or x-coordinate y-coordinate and z-coordinate and we write $M(x, y, z)$.

The vector $\overrightarrow{OM}$ in the Cartesian coordinate system is written as:

$$\boldsymbol{\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}} \quad \text{or} \quad \boldsymbol{\overrightarrow{OM}\begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

In the case of movement, the coordinates x,y and z vary with time: $\overrightarrow{\boldsymbol{OM}}(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{t})\ \vec{\boldsymbol{\imath}} + \boldsymbol{y}(\boldsymbol{t})\ \vec{\boldsymbol{\jmath}} + \boldsymbol{z}(\boldsymbol{t})\vec{\boldsymbol{k}}$

The movement can be done in single axis (straight), Ox for example: $\overrightarrow{\boldsymbol{OM}} = \boldsymbol{x}\vec{\boldsymbol{\imath}}$

In a plane (xOy), $\overrightarrow{OM}$ becomes: $\overrightarrow{\boldsymbol{OM}} = \boldsymbol{x}\vec{\boldsymbol{\imath}} + \boldsymbol{y}\vec{\boldsymbol{\jmath}}$

To plot a point M with (x,y,z) coordinate we should follow the steps:

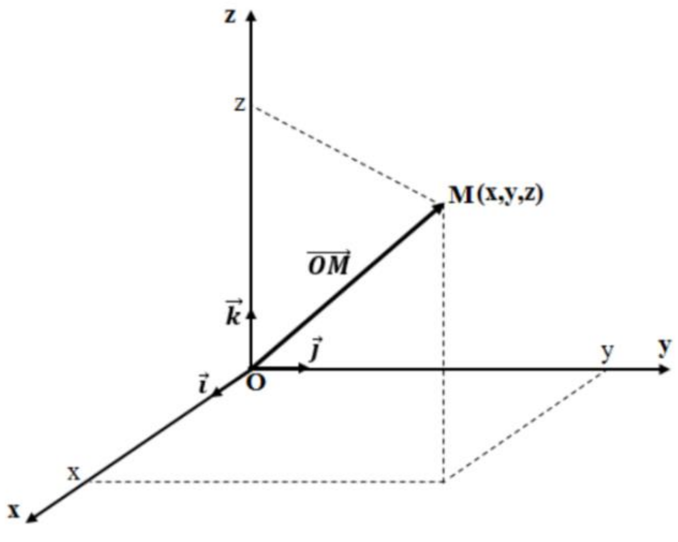
First, we locate the x-coordinate on the x-axis, from this point we draw a dashed line parallel to the y-axis.

Second, locate the y-coordinate on the y-axis and draw a dashed line parallel to the x-axis in manner to obtain a parallelogram.

Third, from the point of intersection of the two dashed lines, we draw a dashed line parallel to the z-axis and then indicate the z-coordinate on this line. This point represents the $M(x, y, z)$ point.

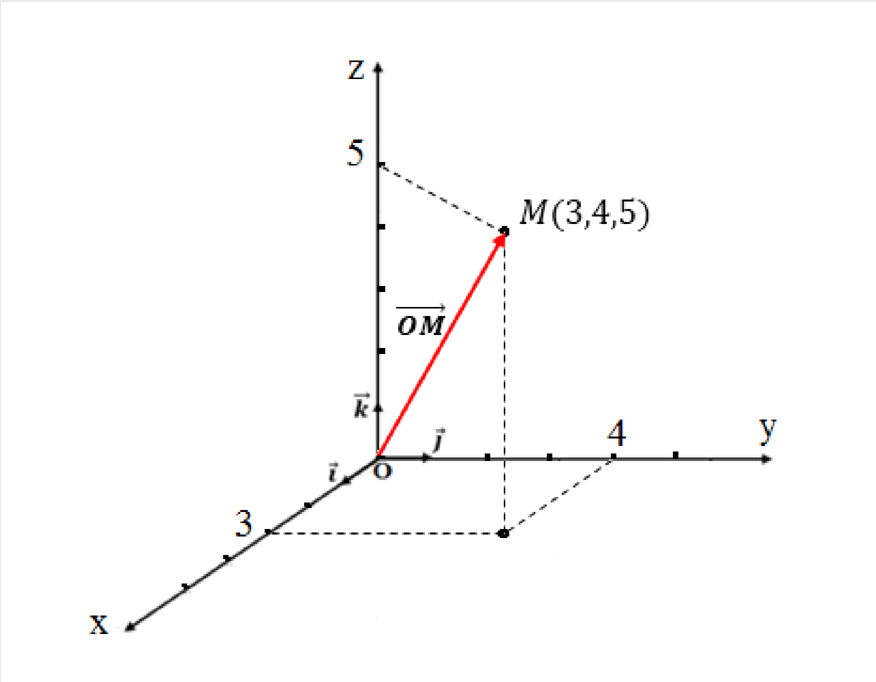
To clarify the plot, we draw a dashed line from the z-coordinate located on the z-axis to the *M* point.

For plotting the vector $\overrightarrow{\boldsymbol{OM}}$, we plot a line from the origin *O* to the *M* point.



Example

Plot the $M(3,4,5)$ *point on xyz-coordinate space and then draw the vector* $\overrightarrow{OM}$:



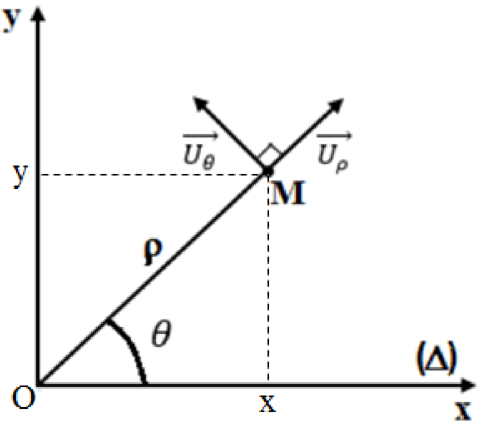
I.4.2 Polar coordinates

If the movement of point M takes place in a plane, the position of this point can be identified by the polar coordinates in which each point is determined by a distance from a reference point (O) called Pole and an angle from a reference direction (a line (Δ)) called polar axis. Generally, we take the Ox axis as polar axis.

The distance from the pole to the point, denoted ρ, is called the radial coordinate or radius and the angle, denoted θ, is called the angular coordinate. Each point *M* in the polar coordinate is described as $M(\rho,\theta)$.

To describe a vector in the polar coordinates system we need to use unit vectors. In the case of polar coordinates there is two directions; the radial direction which is described by a unit radial vector $\overrightarrow{U_\rho}$ and a direction orthogonal to the radial direction described by a unit orthoradial (perpendicular to the radius) vector $\overrightarrow{U_\theta}$. The unit vectors $(\overrightarrow{U_\rho},\overrightarrow{U_\theta})$ are represented in the position of the *M* point, they are perpendicular and they have the same magnitude $\|\overrightarrow{U_\rho}\| = \|\overrightarrow{U_\theta}\| = 1$.

Unlike the coordinate unit vectors in Cartesian coordinates, the unit vectors $(\overrightarrow{U_\rho}, \overrightarrow{U_\theta})$, change their direction depending on the coordinates of the point (ρ, θ).



The relation between the polar coordinates and Cartesian coordinates of point M is: $\begin{cases} \boldsymbol{x = \rho \cos\theta} \\ \boldsymbol{y = \rho \sin\theta} \end{cases}$

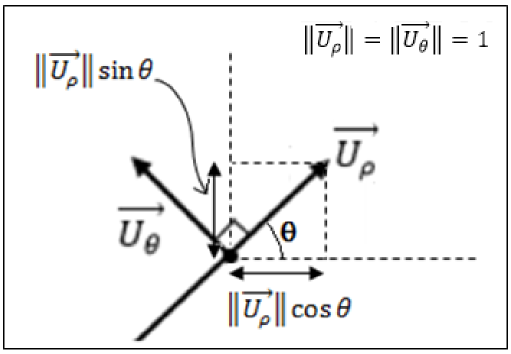
And $\begin{cases} \boldsymbol{\rho = \sqrt{x^2 + y^2}} \\ \boldsymbol{\tan\theta = \frac{y}{x}} \end{cases}$

The relations between polar and Cartesian coordinates unit vectors are:

$\overrightarrow{U_\rho} = \|\overrightarrow{U_\rho}\| \cos\theta\, \vec{\imath} + \|\overrightarrow{U_\rho}\| \sin\theta\, \vec{\jmath}$

Since $\|\overrightarrow{U_\rho}\| = \|\overrightarrow{U_\theta}\| = 1$, we add π/2 to θ to find $\overrightarrow{U_\theta}$.

So: $\begin{cases} \boldsymbol{\overrightarrow{U_\rho} = \cos\theta\, \vec{\imath} + \sin\theta\, \vec{\jmath}} \\ \boldsymbol{\overrightarrow{U_\theta} = -\sin\theta\, \vec{\imath} + \cos\theta\, \vec{\jmath}} \end{cases}$



Because the vector $\overrightarrow{OM}$ is parallel to the radial direction which who wears the $\overrightarrow{U_\rho}$ unit vector, the vector $\overrightarrow{OM}$ is described in the polar coordinates as: $\boxed{\boldsymbol{\overrightarrow{OM} = \rho \overrightarrow{U_\rho}}}$

Examples

1/ The vector $\vec{V}$ *is defined in Cartesian coordinates by:* $\vec{V} = 2\vec{\imath} + 3\vec{\jmath}$ *; we can write the vector in polar coordinates as:* $\vec{V} = \rho\overrightarrow{U_\rho}$

We calculate ρ*:* $\rho = \sqrt{x^2 + y^2} = \sqrt{4+9} = \sqrt{13}$

So the vector $\vec{V}$ *is written as:* $\vec{V} = \sqrt{13}\,\overrightarrow{U_\rho}$.

2/ A point M defined in the polar coordinate as $M\ (3, \frac{\pi}{3})$*, the Cartesian coordinates of M are:*

$x = 3\cos\frac{\pi}{3} = \frac{3}{2}$ *and* $y = 3\sin\frac{\pi}{3} = 3\frac{\sqrt{3}}{2}$

So the point M have the polar coordinates $M(\frac{3}{2}, 3\frac{\sqrt{3}}{2})$.

The derivation of the unit vectors $(\overrightarrow{U_\rho}, \overrightarrow{U_\theta})$

If a point M is in movement with time, the coordinates (ρ, θ) are variable with time and the unit vector are variable with time. We can derivate the unit vectors as follows:

$$\overrightarrow{U_\rho} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} \quad \Rightarrow \quad \frac{d\overrightarrow{U_\rho}}{dt} = \frac{d\cos\theta}{dt}.\frac{d\theta}{d\theta}\vec{\imath} + \frac{d\sin\theta}{dt}.\frac{d\theta}{d\theta}\vec{\jmath} = \frac{d\cos\theta}{d\theta}.\frac{d\theta}{dt}\vec{\imath} + \frac{d\sin\theta}{d\theta}.\frac{d\theta}{dt}\vec{\jmath}$$

$\cos\theta\ (t)$ is a composite function, we must multiply by $\frac{d\theta}{d\theta} = \dot{\theta}$ to obtain:

- $\frac{d\cos\theta}{dt} = \frac{d\cos\theta}{dt}.\frac{d\theta}{d\theta} = \frac{d\cos\theta}{d\theta}.\frac{d\theta}{dt} = -\sin\theta$
- $\frac{d\sin\theta}{dt} = \frac{d\sin\theta}{dt}.\frac{d\theta}{d\theta} = \frac{d\sin\theta}{d\theta}.\frac{d\theta}{dt} = \cos\theta\,.\dot{\theta}$

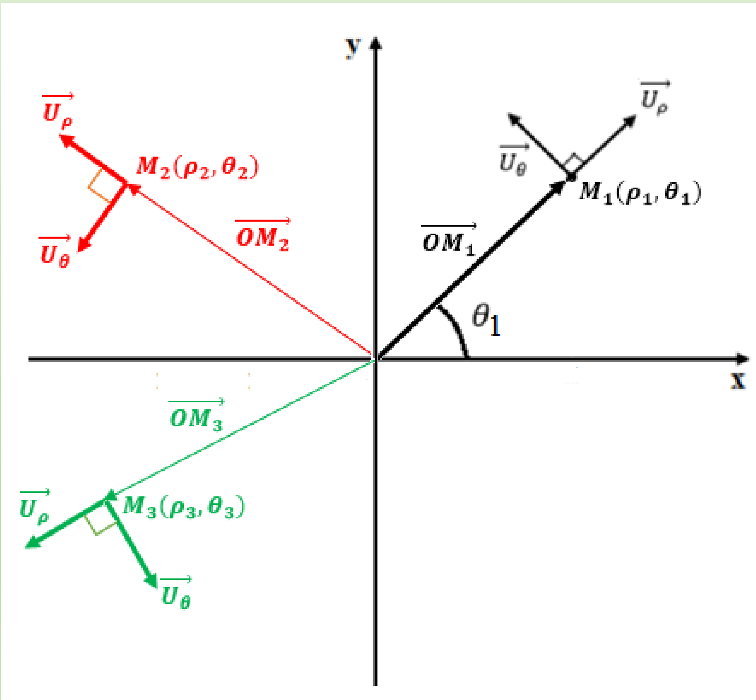
We obtain: $\frac{d\overrightarrow{U_\rho}}{dt} = -\dot{\theta}\sin\theta\,\vec{\imath} + \dot{\theta}\cos\theta\,\vec{\jmath} = \dot{\theta}(-\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath})$

So: $\boxed{\frac{d\overrightarrow{U_\rho}}{dt} = \dot{\theta}.\overrightarrow{U_\theta}}$

Similarly, we find: $\boxed{\frac{d\overrightarrow{U_\theta}}{dt} = -\dot{\theta}.\overrightarrow{U_\rho}}$

Note:

- To draw the unit vectors on any point $M(\rho, \theta)$, we should follow the following steps:
- To draw $\overrightarrow{\boldsymbol{U_\rho}}$: we draw a unit vector from the point M superimposed (parallel) on the vector $\overrightarrow{OM}$.
- To draw $\overrightarrow{\boldsymbol{U_\theta}}$: we draw a unit vector from the point M, perpendicular on $\overrightarrow{U_\rho}$ vector in the counterclockwise direction.



Graphic representation of trajectory

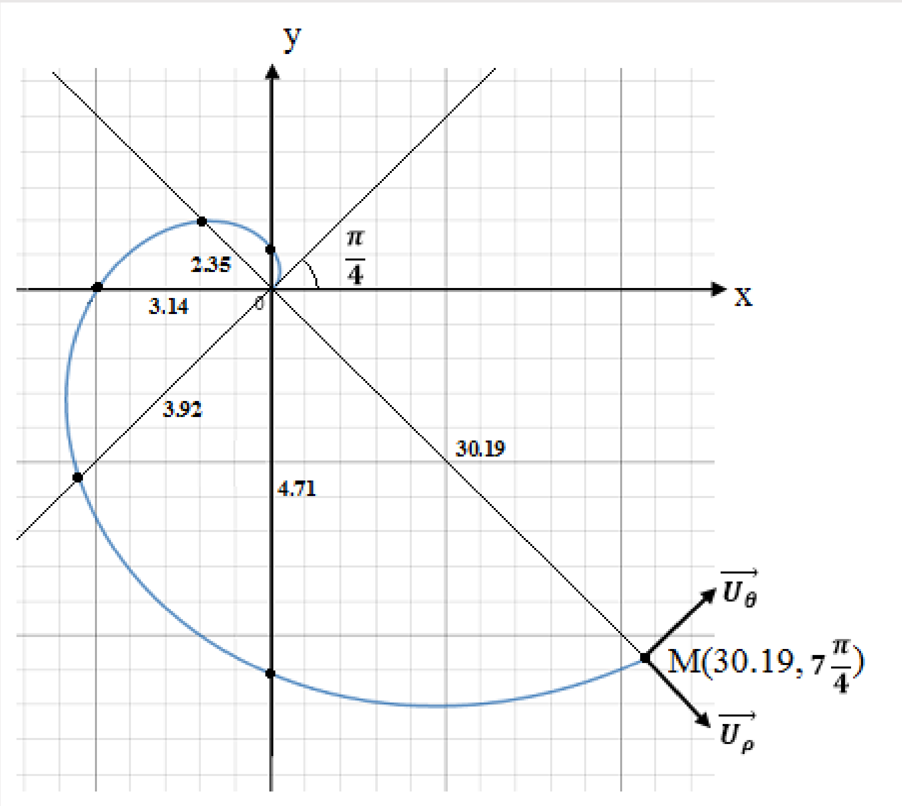
To draw the variation of ρ in function of θ in the polar coordinates system, we must divide the xOy plane into angles. We assign the angle θ with famous angles such as π/6, π/4, π/3, π/2, ..., 2π. Then, we calculate the values of ρ for each angle using the relation between ρ and θ. We arrange these values in a table. Next, we draw all the angles and, on each angle, we plot the points with the values of ρ. Finally, we obtain the trajectory by connecting all points with a line.

Example:

Draw the trajectory $\rho = f(\theta)$ *where:* $\rho = \theta^2$ *from* $\theta = 0$ *to* $\theta = 5\frac{\pi}{2}$

Firstly, we attribute to θ *the values of famous angles (using the radian as angle unit). Then, calculate* ρ *for each angle.*

θ(°)	**0**	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$3\frac{\pi}{4}$	π	$5\frac{\pi}{4}$	$3\frac{\pi}{2}$	$7\frac{\pi}{4}$
θ(rad)	**0**	**0.52**	**0.78**	**1.04**	**1.57**	**2.35**	**3.14**	**3.92**	**4.71**	**5.49**
ρ	**0**	**0.27**	**0.6**	**1.08**	**3.06**	**5.52**	**9.82**	**15.36**	**22.18**	**30.19**



Application

The two points $M_1(3,\frac{\pi}{6})$ *and* $M_2(2,\frac{6\pi}{5})$ *are defined in the polar coordinates. Determine their Cartesian coordinates then represent the vectors* $\overrightarrow{OM_1}$ *and* $\overrightarrow{OM_2}$ *and the unit vectors on the two points.*

Solution:

For $M_1\left(3,\frac{\pi}{6}\right) \Rightarrow \rho = 3 \text{ and } \theta = \frac{\pi}{6}$

$$x = \rho\cos\theta = 3\cos\frac{\pi}{6} = 3\frac{\sqrt{3}}{2} = 2.95$$

$$y = \rho\sin\theta = 3\sin\frac{\pi}{6} = 3.\frac{1}{2} = 1.5$$

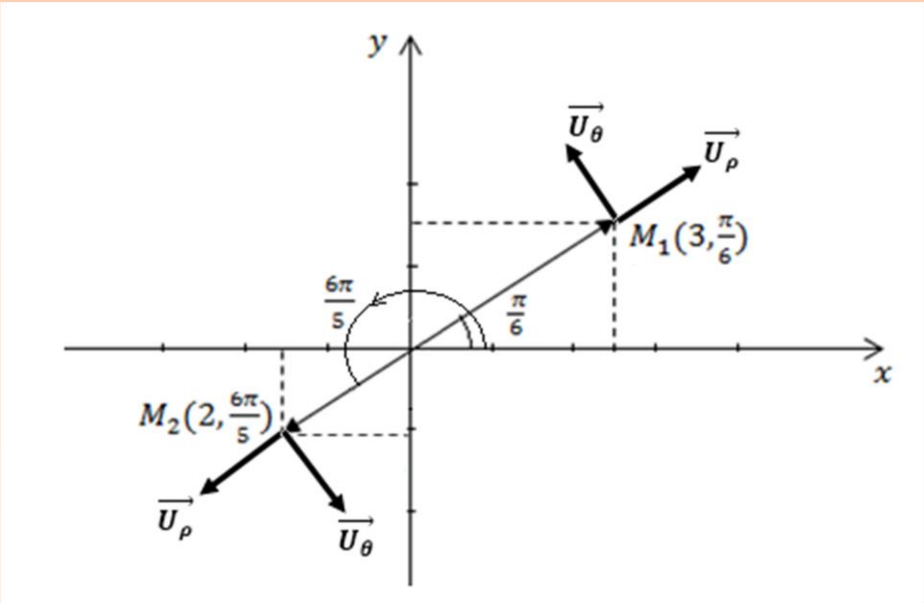
The Cartesian coordinates of M1 point are: $M_1(3\frac{\sqrt{3}}{2},\frac{3}{2})$

For $M_2\left(2,\frac{6\pi}{5}\right) \Rightarrow \rho = 2 \text{ and } \theta = \frac{6\pi}{5}$

$x = -1.61$

$y = -1.17$

The Cartesian coordinates of M_2 *point are* $M_2(-1.618,-1.175)$



I.4.3 Cylindrical coordinates

The cylindrical coordinates result from the association of the polar coordinates and the Oz axis of the Cartesian coordinates, they are characterized by the orthonormal unit vectors $(\overrightarrow{\boldsymbol{U_\rho}}, \overrightarrow{\boldsymbol{U_\theta}}, \vec{\boldsymbol{k}})$ and a point M is defined by the coordinates $M(\rho, \theta, z)$.

The vector $\overrightarrow{OM}$ is written: $\overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM}$

With m is the projection of M on the plane (xOy).

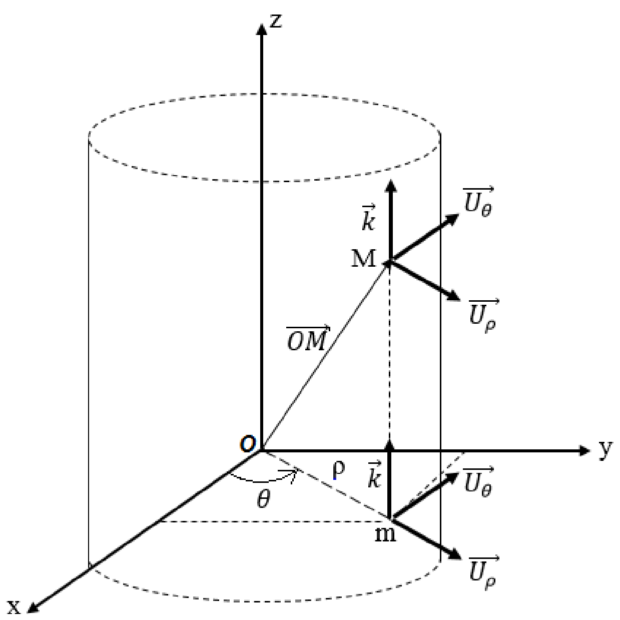
So the vector $\overrightarrow{OM}$ is written as: $\boxed{\overrightarrow{\boldsymbol{OM}} = \boldsymbol{\rho}\overrightarrow{\boldsymbol{U_\rho}} + \boldsymbol{z}\vec{\boldsymbol{k}}}$

With: $\begin{cases} 0 \leq \rho < +\infty \\ -\infty < z < +\infty \end{cases}$

and $\boxed{\left\|\overrightarrow{\boldsymbol{OM}}\right\| = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{z}^2}}$

The relations between the cylindrical (ρ, θ, z) and Cartesian (x, y, z) coordinates are:

$\begin{cases} \boldsymbol{\rho} = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2} \\ \boldsymbol{\theta} = \mathbf{arctan}\,\frac{\boldsymbol{y}}{\boldsymbol{x}} \\ \mathbf{z} = \mathbf{z} \end{cases}$; and $\begin{cases} \boldsymbol{x} = \boldsymbol{\rho}\cos\boldsymbol{\theta} \\ \boldsymbol{y} = \boldsymbol{\rho}\sin\boldsymbol{\theta} \\ \boldsymbol{z} = \boldsymbol{z} \end{cases}$



Examples

1/ The Cartesian coordinates of M point are: $M\ (2{,}3{,}4)$, *it's cylindrical coordinates are:*

$$\begin{cases} \rho = \sqrt{\rho^2 + z^2} = \sqrt{4+9} = \sqrt{13} \\ \theta = arctan\frac{y}{x} = arctan\frac{3}{2} = 0{,}98\ rad \\ z = 4 \end{cases} \quad \text{So:} \quad M(\sqrt{13}, 0{,}98rad, 4)$$

2/ The cylindrical coordinates of M' point are: $M'\left(3, \frac{\pi}{3}, 6\right)$, *it's Cartesian coordinates are:*

$$\begin{cases} x = \rho\cos\theta = 3\cos\frac{\pi}{3} = \frac{3}{2} \\ y = \rho\sin\theta = 3\sin\frac{\pi}{3} = 3\frac{\sqrt{3}}{2} \\ z = z = 6 \end{cases} \quad \text{So:} \quad M'\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 6\right)$$

I.4.4 Spherical coordinates

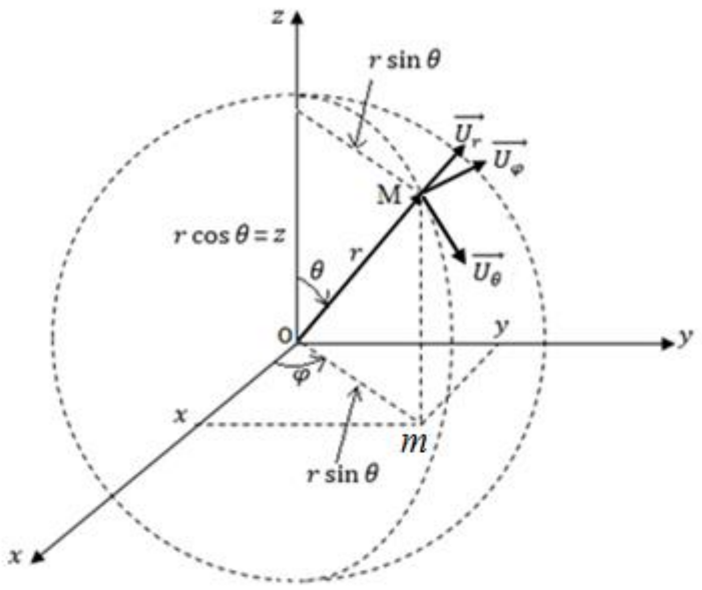
The Spherical coordinates mainly used in three-dimensional systems. The coordinates of a point M are denoted as $M(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi})$.

Where: r is the distance between the origin of the reference O and the point M.

θ is the angle between the vector $\overrightarrow{OM}$ and the Oz axis.

φ is the angle between the vector $\overrightarrow{Om}$ and the Ox axis (m is the projection of M on the xOy plane).

The unit vectors of the spherical coordinates are: $(\overrightarrow{U_r}, \overrightarrow{U_\theta}, \overrightarrow{U_\varphi})$.



$\overrightarrow{U_r}$ along $\overrightarrow{OM}$, $\overrightarrow{U_\theta}$ perpendicular on $\overrightarrow{U_r}$ in the direction of increase of θ angle and $\overrightarrow{U_\varphi}$ is perpendicular on $\overrightarrow{U_r}$ and $\overrightarrow{U_\theta}$ in the direction of increase of φ angle.

Because $\overrightarrow{OM}$ is parallel to the $\overrightarrow{U_r}$ unit vector, the vector $\overrightarrow{OM}$ is written as: $\boxed{\overrightarrow{\boldsymbol{OM}} = \boldsymbol{r}\,\overrightarrow{\boldsymbol{U_r}}}$

With $\begin{cases} \boldsymbol{0 \le r \le +\infty} \\ \boldsymbol{0 \le \theta \le \pi} \\ \boldsymbol{0 \le \varphi \le 2\pi} \end{cases}$ and $\|\overrightarrow{\boldsymbol{OM}}\| = \boldsymbol{r}$

The relations between the Cartesian (x, y, z) and the spherical (r, θ, φ) coordinates are:

$$\begin{cases} \boldsymbol{r = \|\overrightarrow{OM}\| = \sqrt{x^2 + y^2 + z^2}} \\ \boldsymbol{\theta = \arctan \frac{\sqrt{x^2+y^2}}{z}} \\ \boldsymbol{\varphi = \arctan \frac{y}{x}} \end{cases} \quad ; \quad \begin{cases} \boldsymbol{x = \rho \cos\varphi = r \sin\theta \cos\varphi} \\ \boldsymbol{y = \rho \sin\varphi = r \sin\theta \sin\varphi} \\ \boldsymbol{z = r\cos\theta} \end{cases} \text{ with } \boldsymbol{\rho = r\sin\theta}$$

Example

1/ The Cartesian coordinates of M point are: $M\ (4{,}2{,}3)$, *it's spherical coordinates are:*

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4^2 + 2^2 + 3^2} = \sqrt{29} \\ \theta = arctan\frac{\sqrt{x^2+y^2}}{z} = arctan\frac{\sqrt{4^2+2^2}}{3} = 0{,}98\,rad \text{ So: } M(\sqrt{29}, 0{,}98\,rad, 0{,}46\,rad) \\ \varphi = arctan\frac{y}{x} = arctan\frac{2}{4} = 0{,}46\,rad \end{cases}$$

2/ The spherical coordinates of M' point are: $M'\left(3, \frac{\pi}{6}, \frac{\pi}{4}\right)$, *it's Cartesian coordinates are:*

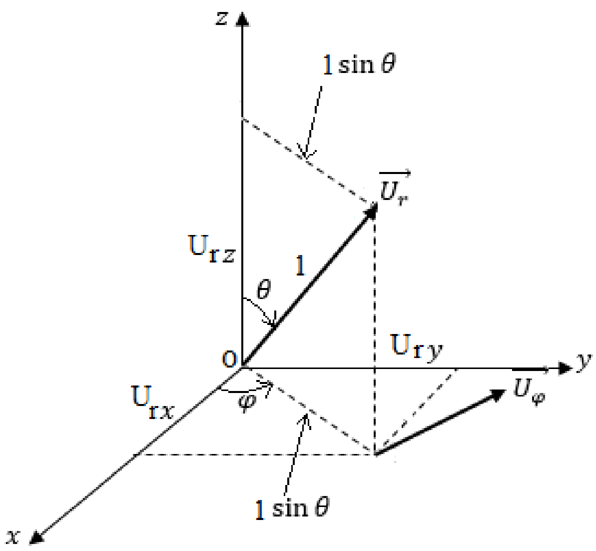
$$\begin{cases} x' = r\sin\theta\cos\varphi = 3.sin\frac{\pi}{6}.cos\frac{\pi}{4} = \frac{3\sqrt{2}}{4} \\ y' = r\sin\theta\sin\varphi = 3.sin\frac{\pi}{6}.sin\frac{\pi}{4} = \frac{3\sqrt{2}}{4} \quad \text{So: } M'(\frac{3\sqrt{2}}{4}, \frac{3\sqrt{2}}{4}, \frac{3\sqrt{3}}{2}). \\ z' = r\cos\theta = 3.cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2} \end{cases}$$

The relation between spherical and* Cartesian *unit vectors

We have: $\overrightarrow{U_r} = U_{rx}\vec{\imath} + U_{ry}\vec{\jmath} + U_{rz}\vec{k}$

$$\begin{cases} U_{rx} = 1.\sin\theta\,.\cos\varphi \\ U_{ry} = 1.\sin\theta\,.\sin\varphi \\ U_{rz} = 1.\cos\theta \end{cases}$$

So: $\overrightarrow{U_r} = sin\,\theta\,.cos\,\varphi\,.\vec{\imath} + sin\,\theta\,.sin\,\varphi\,.\vec{\jmath} + \cos\theta\,\vec{k}$



$\overrightarrow{U_\theta}$ is perpendicular on $\overrightarrow{U_r}$, we can found it by adding π/2 to θ in the formula of $\overrightarrow{U_r}$.

$$\overrightarrow{U_\theta} = \sin(\theta + \frac{\pi}{2}).\cos\varphi.\vec{\imath} + \sin(\theta + \frac{\pi}{2}).\sin\varphi.\vec{\jmath} + \cos(\theta + \frac{\pi}{2})\,\vec{k}$$

$$\overrightarrow{U_\theta} = \cos\theta.\cos\varphi.\vec{\imath} + \cos\theta.\sin\varphi.\vec{\jmath} - \sin\theta\,\vec{k}$$

Concerning $\overrightarrow{U_\varphi}$, we calculate it as in the case of the polar coordinates:

$$\overrightarrow{U_\varphi} = -\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath}$$

So we have: $\begin{cases} \boldsymbol{\overrightarrow{U_r} = \sin\theta.\cos\varphi.\vec{\imath} + \sin\theta.\sin\varphi.\vec{\jmath} + \cos\theta\,\vec{k}} \\ \boldsymbol{\overrightarrow{U_\theta} = \cos\theta.\cos\varphi.\vec{\imath} + \cos\theta.\sin\varphi.\vec{\jmath} - \sin\theta\,\vec{k}} \\ \boldsymbol{\overrightarrow{U_\varphi} = -\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath}} \end{cases}$

The derivation of unit vectors $(\overrightarrow{U_r}, \overrightarrow{U_\theta}, \overrightarrow{U_\varphi})$:

- $\frac{d\overrightarrow{U_r}}{dt} = \dot{\theta}\overrightarrow{U_\theta} + \dot{\varphi}.\sin\theta\,\overrightarrow{U_\varphi}$
- $\frac{d\overrightarrow{U_\theta}}{dt} = -\dot{\theta}\overrightarrow{U_r} + \dot{\varphi}.\cos\theta\,\overrightarrow{U_\varphi}$
- $\frac{d\overrightarrow{U_\varphi}}{dt} = -\,\dot{\varphi}(\sin\theta\,\overrightarrow{U_r} + \cos\theta\,\overrightarrow{U_\theta})$

I.5 Elementary displacement in different coordinates

The elementary displacement vector $\overrightarrow{dl}$ is the vector between two very close points.

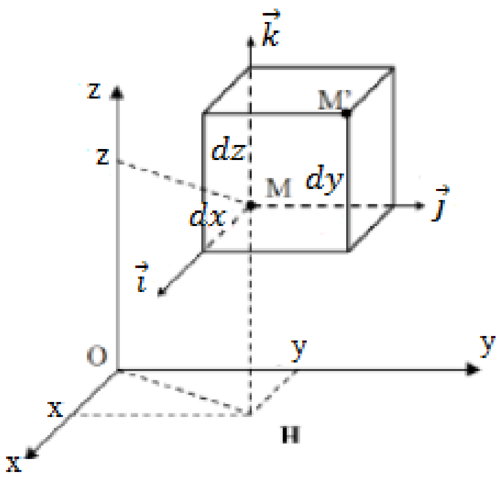
$$\overrightarrow{dl} = \lim_{M' \to M} \overrightarrow{MM'} = d\overrightarrow{OM}$$

We find this vector by passing over the main axes of the chosen reference frame.

Cartesian coordinates

We carry out an infinitesimal variation of dx on Ox axis, dy on Oy axis and dz on Oz axis we find:

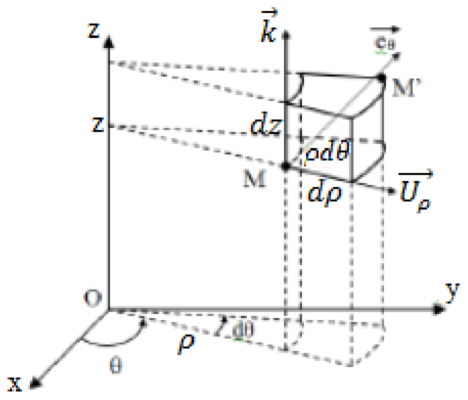
$$\overrightarrow{dl} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$$



Cylindrical Coordinates

We carry out an infinitesimal variation of $d\rho$ on $\overrightarrow{U_\rho}$, $\rho.d\theta$ on $\overrightarrow{U_\theta}$ and dz on Oz we find:

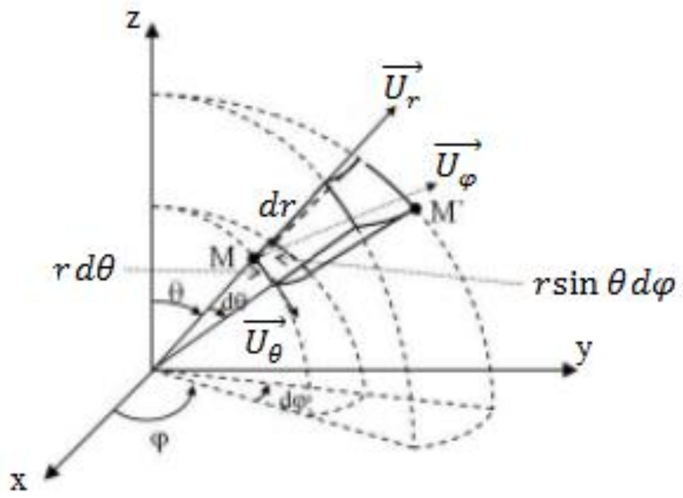
$$\overrightarrow{dl} = d\rho\,\overrightarrow{U_\rho} + \rho.d\theta\,\overrightarrow{U_\theta} + dz\,\vec{k}$$



Spherical coordinates

We carry out an infinitesimal variation of dr on $\overrightarrow{U_r}$, $r.d\theta$ on $\overrightarrow{U_\theta}$ and $r.\sin\theta\,.d\varphi$ on $\overrightarrow{U_\varphi}$ we find:

$$\overrightarrow{dl} = \overrightarrow{MM'} = dr\,\overrightarrow{U_r} + r.d\theta\,\overrightarrow{U_\theta} + r.\sin\theta\,.d\varphi\,\overrightarrow{U_\varphi}$$



I.5 Error calculation

When measuring the same physical amount several times, a discrepancy is observed in the results obtained, it is difficult to perform rigorously accurate measurements and this is due to an error in the measurement. This error is always affected by limitations of instruments, experimental conditions, and human factors. Error calculation, is therefore essential to quantify the reliability and accuracy of measured quantities.

I.5.1 Nature of the errors

To study physical phenomena, we rely in most cases on measurements that are characterized by the absence of accurate determination of the measured value resulting from error. There are three types of error.

Systematic error is a consistent and repeatable error caused by measurement system and causes all measurements to deviate from the true value in the same predictable direction. This type of error typically origin from flaws in the instrument (like calibration), the experimental method (environmental conditions), or the observer. To avoid this error, this error must be identified and corrected through methods like calibrate the instrument, or applying a calculated correction factor based on a thorough analysis of the error's source.

Random error: is the unpredictable variation in measurement, often described as statistical noise. It arises from uncontrollable minor influences, such as environmental changes, instrumental sensitivity which cause individual readings of the measurement randomly near the true value. Although these fluctuations affect the precision of repeated measurements, they do not consistently change the average in one direction, meaning the overall accuracy can be preserved by taking the mean of sufficient data points.

Accidental error: also known as a gross error refers to an abnormal deviation in a measurement, typically caused by improper use of the instrument or faulty handling, or a temporary malfunction of the measurement device. we can avoid this error through careful operation and routine instrument checks.

Random, systematic and accidental errors describe the nature and source of an error, mathematically the way to quantify the size or the impact of an error, regardless of its type is using absolute and relative errors.

Absolute error and absolute uncertainly

The absolute error of a measurement is the positive difference between the measured (or experimental) value and the accepted reference value that we have a good reason to consider as true. It quantifies the magnitude of the measurement's deviation from the reference without regard to direction, providing a direct numerical estimate of accuracy in the same units as the measured quantity.

If: G_m represents the measured value of the quantity and G the exact theoretical value of the same quantity, the absolute error δ is:

$$\delta G = G_m - G$$

As the exact value of the quantity to be measured is unknown, it is necessary evaluate an upper limit of the absolute error which is none other than the absolute uncertainty noted:

$$\Delta G = sup(|\delta X|)$$

Relative error and relative uncertainly

The relative error it is defined as the ratio of the absolute error and the true value of the quantity to be measured. It is a dimensionless quantity it is expressed generally in percentage (%), it can take either positive or negative values depending on whether the measured value is greater or smaller than the true value.

$$\frac{\delta G}{G} = \frac{G_m - G}{G}$$

In practice, we often cannot know the absolute or relative error precisely because we do not know the true value. This leads us directly to the concept of uncertainty. If the exact value of the quantity is inaccessible, we will take the upper limit of the relative error which is none other than the relative uncertainty $\frac{\Delta G}{G}$, it is expressed in percentage (%).

The value adopted is equal to the measured value followed by the evaluation of the absolute uncertainty:

$$G = G_m \pm \Delta G \ (unit)$$

I.5.2 Uncertainty calculation

Perfect differential method

Let the quantity G be related to the independent values $G(x, y, z)$, we calculate absolute uncertainty ΔG and relative uncertainty $\Delta G/G$ starting from Δx, Δy and Δz. For measurements x, y, z where Δx, Δy and Δz are small compared to its measurement and thus we apply the differential calculus rule:

$$\Delta G = \sup|dG|$$

Where the differential of G is: $dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial z}dz$

$$|dG| = \left|\frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}\,\mathrm{dy} + \frac{\partial G}{\partial z}\,\mathrm{dz}\right| = \left|\frac{\partial G}{\partial x}\right||dx| + \left|\frac{\partial G}{\partial y}\right||dy| + \left|\frac{\partial G}{\partial z}\right||dz|$$

$$\Delta G = \sup|dG| = \left|\frac{\partial G}{\partial x}\right||dx| + \left|\frac{\partial G}{\partial y}\right||dy| + \left|\frac{\partial G}{\partial z}\right||dz| = \left|\frac{\partial G}{\partial x}\right|\Delta x + \left|\frac{\partial G}{\partial y}\right|\Delta y + \left|\frac{\partial G}{\partial z}\right|\Delta z$$

The absolute uncertainly is: $\frac{\Delta G}{G} = \left|\frac{X\partial G}{G\partial x}\right|\frac{\Delta x}{x} + \left|\frac{y\partial G}{G\partial y}\right|\frac{\Delta y}{y} + \left|\frac{z\partial G}{G\partial z}\right|\frac{\Delta z}{z}$

The absolute uncertainty on (f) is generally expressed in the form:

$$\Delta G = \left|\frac{\partial G}{\partial x}\right|\Delta x + \left|\frac{\partial G}{\partial y}\right|\Delta y + \left|\frac{\partial G}{\partial z}\right|\Delta z$$

a- Logarithmic differentiation

To calculate the relative uncertainty on the function G using the logarithmic differential method, the following steps should be followed:

1. Introduce the logarithmic function to the function G: $lnG = \ln G(x, y, z)$

2. Calculate: $d(lnG) = \frac{dG}{G}$

3. Deduce relative uncertainly on G: $\frac{dG}{G} \leq \frac{\Delta G}{G}$

Set of exercises

(*Units and vectors, coordinate systems*)

Set of exercises (Units and vectors, coordinate systems)

Exercise 1:

1/ Show that the expression $\boldsymbol{v = v_0 + at}$ is dimensionally correct, where v and v_0 represent velocities, a is acceleration, and t is time.

2/ The period of a simple pendulum is $\boldsymbol{T = 2\pi\sqrt{\frac{l}{g}}}$. Show that this equation is dimensionally correct.

Exercise 2:

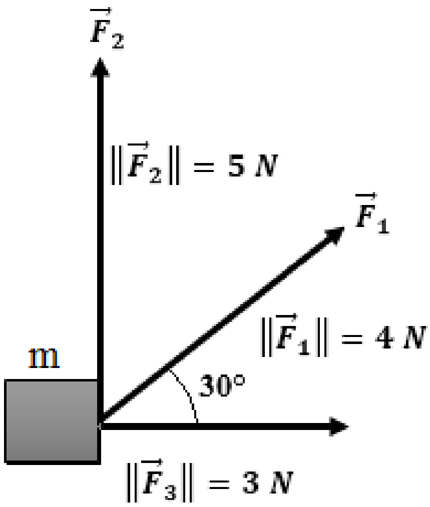
Find the magnitude of the vector $\vec{C}$ that satisfies the equation: $2\vec{A} - 6\vec{B} + 3\vec{C} = 2\vec{j}$,
where $\vec{A} = \vec{\imath} - 2\vec{k}$; $\vec{B} = -\vec{j} + \frac{1}{2}\vec{k}$

Exercise 3:

Given two vectors:
Vector $\overrightarrow{V_1}$ with: $\|\overrightarrow{V_1}\| = 10\ m$, angle direction 30°;
Vector $\overrightarrow{V_2}$ with: $\|\overrightarrow{V_2}\| = 20\ m$, angle direction 60°.
1/ Draw the two vectors in a xOy plane.
2/ Determine the components of the two vectors.
3/Determine their sum then calculate the magnitude and the angle direction of the resultant vector, draw it.

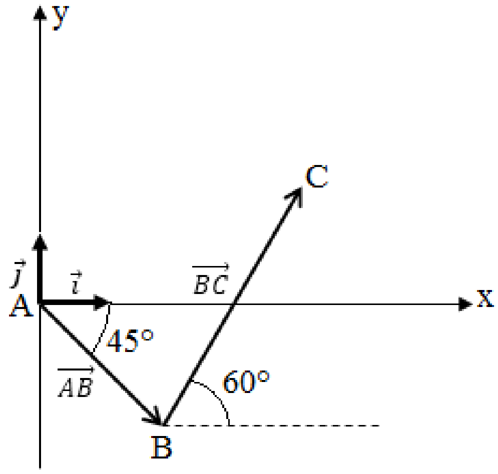
Exercise 4:

Three forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ are acting on an object m as shown in the following figure. Find the resultant force (the vector) and draw it.



Exercise 5:

Vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are shown in the following figure:



Where: $\|\overrightarrow{AB}\| = 25\ m$ and $\|\overrightarrow{BC}\| = 40\ m$

1/ Find the vector components of vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

2/ Find the vector $\overrightarrow{AC}$, then calculate his magnitude and his angle direction.

Exercise 6:

Given three vectors $\overrightarrow{V_1}$ ،$\overrightarrow{V_2}$ and $\overrightarrow{V_3}$ where:

$\overrightarrow{V_1} = 3\vec{\imath} - 4\vec{\jmath} + \vec{k}$; $\overrightarrow{V_2} = 5\vec{\imath} - \vec{\jmath} + 3\vec{k}$ and

$\overrightarrow{V_3} = 2\vec{\imath} + 3\vec{\jmath} - 4\vec{k}$

1/ Calculate the magnitude of these vectors.

2/ Find the components of $\vec{A}$ and $\vec{B}$ vectors, where;

$\vec{A} = \overrightarrow{V_1} + \overrightarrow{V_2} + \overrightarrow{V_3}$ and $\vec{B} = 2\overrightarrow{V_1} - \overrightarrow{V_2} + \overrightarrow{V_3}$

3/ Find $\vec{u}_C$ the unit vector for vector $\vec{C}$, where:

$\vec{C} = \overrightarrow{V_1} + \overrightarrow{V_3}$

4/ Calculate the scalar product $(\overrightarrow{V_1}.\overrightarrow{V_3})$ then conclude the angle between the two vectors.

Exercise 7:

Given three vectors: $\vec{A} = 2\vec{\imath} - \vec{\jmath} + \vec{k}$,

$\vec{B} = \vec{\imath} + \vec{\jmath} - 2\vec{k}$ *and* $\vec{C} = x\vec{\imath} + \vec{\jmath} + y\vec{k}$

1/ Find the value of x and y in order that $\vec{A}//\vec{C}$.

2/ Find the value of x and y in order that $\vec{B}//\vec{C}$.

3/ Find the value of x and y in order that $\vec{C} \perp \vec{A}$ and $\vec{C} \perp \vec{B}$.

For these values of x and y, find the volume of the parallelepiped based on the three vectors.

4/ For $x = -1$ *and* $y = -2$ check the relation:

$\vec{A}\wedge(\vec{B}\wedge\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$

Exercise 8:

1/ In the Cartesian coordinates space $(O,\vec{\imath},\vec{\jmath})$, given two points $M_1(5,4)$ and $M_2(-3,3)$, draw M_1 and M_2 then plot:

- The vector $\overrightarrow{V_1} = \vec{\imath} + 2\vec{\jmath}$ in point of application M_1.
- The vector $\overrightarrow{V_2} = -2\vec{\imath} - \vec{\jmath}$ in point of application M_1.

2/ In the Cartesian coordinates space $(O,\vec{\imath},\vec{\jmath},\vec{k})$, given two points $M_3(3,-2,3)$ and $M_4(0,5,3)$, plot M_3 and M_4 then calculate and draw the vector sum $\overrightarrow{OM_3}, \overrightarrow{OM_3}$.

Exercise 9:

In the Polar coordinates space $(O,\overrightarrow{U_\rho},\overrightarrow{U_\theta})$, given two points $P_1(3,\frac{\pi}{3})$ and $P_2(4,\frac{\pi}{4})$.

1/ Find the components of these two points in Cartesian coordinate system.

2/ Represent P_1 and P_2 in Polar coordinates space then draw:

a- The unit vectors $\overrightarrow{U_\rho}$ and $\overrightarrow{U_\theta}$ in each point.

b- The vector $\overrightarrow{V_1} = 2\overrightarrow{U_\rho} + 3\overrightarrow{U_\theta}$ in the point of application P_1.

c- The vector $\overrightarrow{V_2} = 3\overrightarrow{U_\theta}$ in the point of application P_2.

3/ For the two points P_1 and P_2, find, in the Cartesian coordinates system, the components of unit vectors in each point.

Exercise 10:

In the Polar coordinates system, we consider the relation: $\rho(\theta) = 2\theta + 3$

1/ Find the set of points defined by the coordinates M(ρ,θ) when the angle θ changes in the range [0,π].

2/ Plot the curve $\rho = \rho(\theta)$ passing on M points in the same range.

1/ Write the expression of the vector $\overrightarrow{OM}$, then calculate the derivatives $\frac{d\overrightarrow{OM}}{d\theta}$ and $\frac{d^2\overrightarrow{OM}}{d\theta^2}$.

2/ For: $\theta = \frac{\pi}{4}$ draw both of the unit vectors $(\overrightarrow{U_\rho},\overrightarrow{U_\theta})$ and $\overrightarrow{OM}$.

Exercise 11:

Point P has cylindrical coordinates $(5,\frac{\pi}{4},6)$:

1/ Plot P then the vector $\overrightarrow{OP}$ then draw the unit vectors of cylindrical coordinates system on the point P.

2/ Find the coordinates of P point the Cartesian coordinates system.

Exercise 12:

1/ Plot the point P with spherical coordinates $(6,\frac{\pi}{6},\frac{\pi}{3})$ and draw, on this point, the unit vectors of spherical coordinates.

2/ Convert the Cartesian coordinates $(-1,1,\sqrt{6})$

to both spherical and cylindrical coordinates.

Chapter II:

Kinematics of a material point

Chapter II: Kinematics of a material point

II.1 Introduction

Kinematics is the study of the movement of objects (bodies) of small speeds, compared to the speed of light ($c = 3.108\, m.s^{-1}$), without considering the forces that cause the movement. The study of the movement requires determining a reference on which we define a system of coordinates and a time scale appropriate for the physical problem.

The purpose of this chapter is to define the motion characteristics (position vector, velocity, and acceleration) of a moving body in different coordinate systems, such as Cartesian, polar, cylindrical, and spherical coordinates.

The Definition of a Material Point

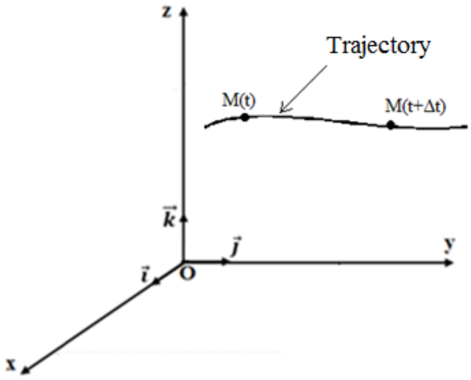
In physics, an object (a body) in motion is represented by a material point (or particle) that has zero dimensions. The dimensions of the solid are negligible, and the rotation of the solid on itself is also negligible. In this course, a material point is denoted by the letter M.

A material point is an idealized object that is used in physics to simplify the analysis of motion. For example, we consider a plane as a material point; we follow its location, speed, and acceleration regardless of its dimensions.

The position of an object in movement, in the case of the Cartesian coordinate system, is defined entirely by the three coordinates of a single point belonging to the object (x, y, z) which vary over time x(t), y(t), and z(t). These three equations represent the equations of motion or parametric equations, where time is the parameter.

Point Trajectory (or point track) is the line along which the point M moves in space. The point track can be a straight line, a curve, or a combination of both. It can use to predict the future motion of the point.

The trajectory is described by the relation between the coordinates of the point M, where the parameter of time (t) does not appear in this relation. In the case of the Cartesian coordinates, we obtain a function of form $f(x, y, z) = 0$, which represents the equation of the trajectory.



Example

The equations of motion of a material point are: $\begin{cases} x(t) = 4t \\ y(t) = 2t^2 - 3 \end{cases}$

We have: $x = 4t \quad \Rightarrow \quad t = \frac{x}{4}$

We replace the value of (t) in the relation of (y) we find: $y = \frac{1}{8}x^2 - 3.$

The trajectory equation of the motion of the point material is: $y(x) = \frac{1}{8}x^2 - 3.$

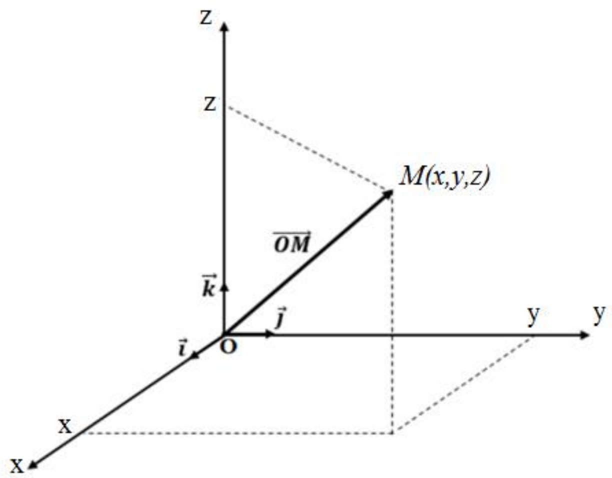
II.2 Kinematic quantities in the different coordinate systems

II.2.1 Kinematic quantities in Cartesian coordinates

A material point in the Cartesian coordinate (three dimensions) is located with the coordinates (x,y,z). If the point is in motion, the variables x, y, and z are functions of time (t);

$$\boldsymbol{x = x(t), y = y(t), z = z(t)}$$

The **position vector**, from the origin of the coordinate system to the point M, is defined as the vector $\overrightarrow{OM}(t)$ or $\vec{r}(t)$: $\boldsymbol{\vec{r}(t) = \overrightarrow{OM} = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}}$



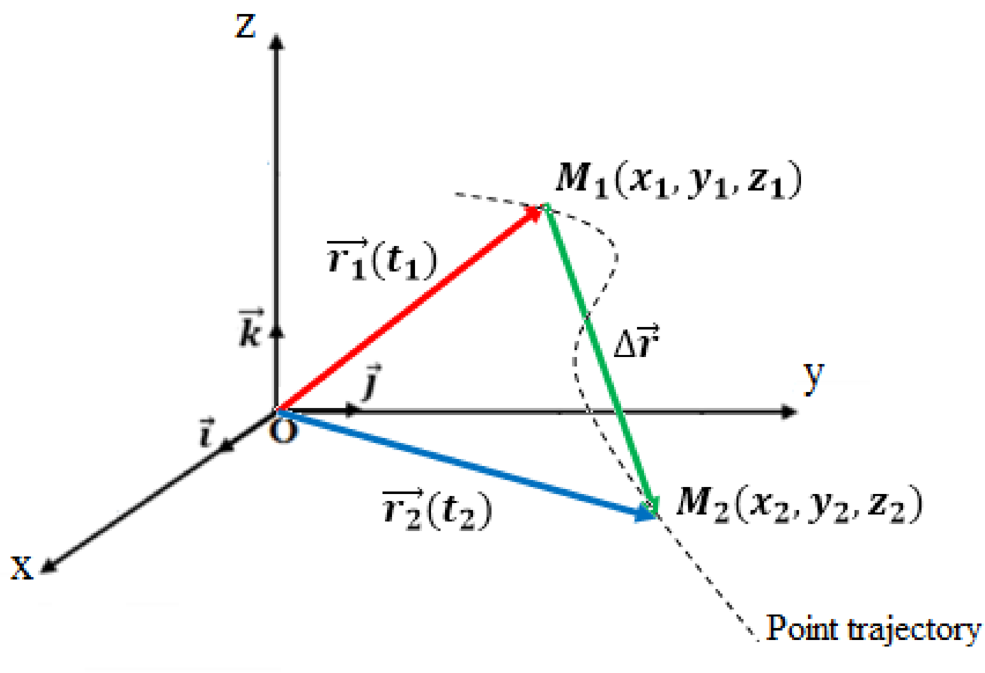
At the time t_1 , the material point is located at the point $M_1(x_1, y_1, z_1)$ with vector position $\vec{r_1}(t_1)$:

$$\vec{r_1}(t_1) = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$

After the period of time t_2 , the material point will be located at the point $M_2(x_2, y_2, z_2)$ with vector position $\vec{r_2}(t_2)$:

$$\vec{r_2}(t_2) = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

The d**isplacement vector** is the vector $\Delta\vec{r}$ that represents the vector from the point M_1 to the point M_2. We find the displacement vector by subtracting vector positions: $\Delta\vec{r} = \vec{r_2}(t_2) - \vec{r_1}(t_1)$



Note:

In the Cartesian coordinates, there are several types of trajectory equations, for example:

- In the case of straight motion, the trajectory equation takes the form of a straight line $y = ax + b$.
- In the case of parabola, the trajectory equation takes the form of a parabola equation: $y = ax^2 + bx + c$.
- In the case of a circular motion, the trajectory equation takes the form of a circle equation: $(x - a)^2 + (y - b)^2 = R^2$

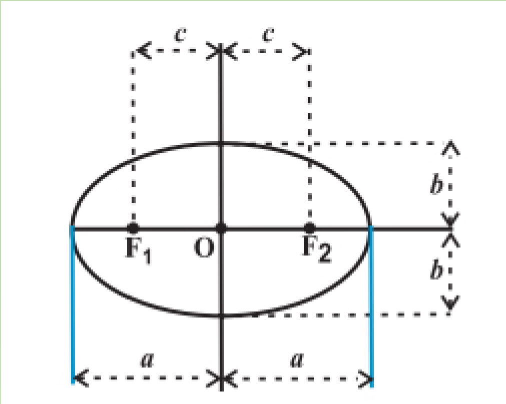
  So R is the radius of the circle and (a, b) the coordinates of the circle's center.

  If the circle's center is $(0,0)$, the trajectory equation becomes $x^2 + y^2 = R^2$.
- In the case of an ellipse, the trajectory equation takes the form (standard form): $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(a) is the length of the semi-major axis, and (b) is the length of the Semi-minor axis (see figure).

If the center of the ellipse differ of (0,0), the equation of the ellipse takes the form:

$\frac{(x-a)^2}{a^2} + \frac{(y-b)^2}{b^2} = 1$, the point with components (a, b) is the center of the ellipse.



Horizontal ellipse.

Example

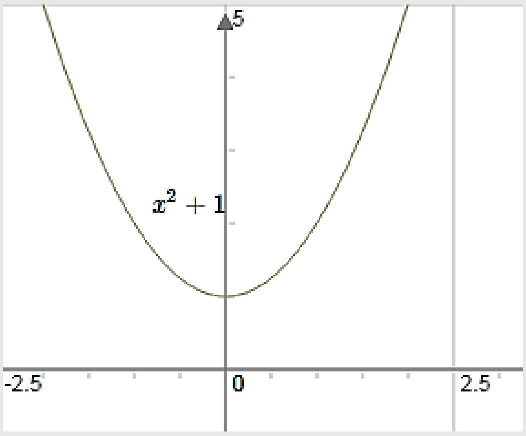
Find the trajectory equation of a material point in motion with parametric equations:

1/ $\begin{cases} x = 2t \quad \ldots\ldots(1) \\ y = 4t^2 + 1 \ldots\ldots(2) \end{cases}$

From equation (1): $t = \frac{x}{2}$

Substituting the value of t in equation (2)

$$y = 4\left(\frac{x}{2}\right)^2 + 1 \Longrightarrow y(x) = x^2 + 1$$



2/ $\begin{cases} x = R + R\cos 2\theta \\ y = R\sin 2\theta \end{cases}$

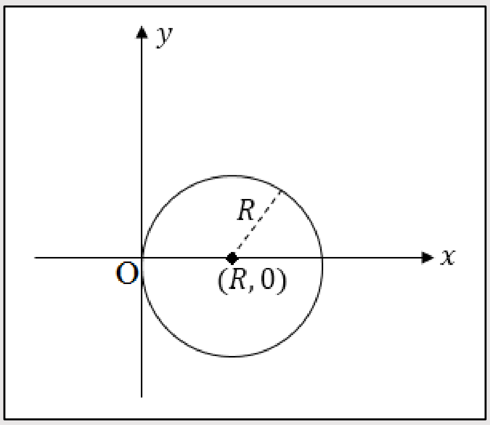
From these equations we extract $(\cos 2\theta)$ *and* $(\sin 2\theta)$: $\begin{cases} \cos 2\theta = \frac{x-R}{R} \\ \sin 2\theta = \frac{y}{R} \end{cases}$

Squaring both sides of the two equations, we find: $\begin{cases} \cos^2 2\theta = \frac{(x-R)^2}{R^2} \\ \sin^2 2\theta = \frac{y^2}{R^2} \end{cases}$

Adding the two equations: $\cos^2 2\theta + \sin^2 2\theta = \frac{(x-R)^2}{R^2} + \frac{y^2}{R^2} \Longrightarrow 1 = \frac{(x-R)^2}{R^2} + \frac{y^2}{R^2}$

$\Longrightarrow (x-R)^2 + y^2 = R^2$

This equation represents the circle's equation of radius R and $(R, 0)$ *center.*



Velocity Vector

The speed of an object is the scalar quantity defined as the rate of change of distance with time. However, **the velocity** is the vector quantity defined as the rate of change of displacement with time. The speed is measured in meters per second (m/s or m.s^{-1}).

The average speed (v_a) of an object in motion is defined by the ratio total displacement between two points divided by the time taken to travel between them:

$$v_{av} = \frac{\textbf{\textit{distance between two points}}}{\textbf{\textit{Elapsed time between two points}}}$$

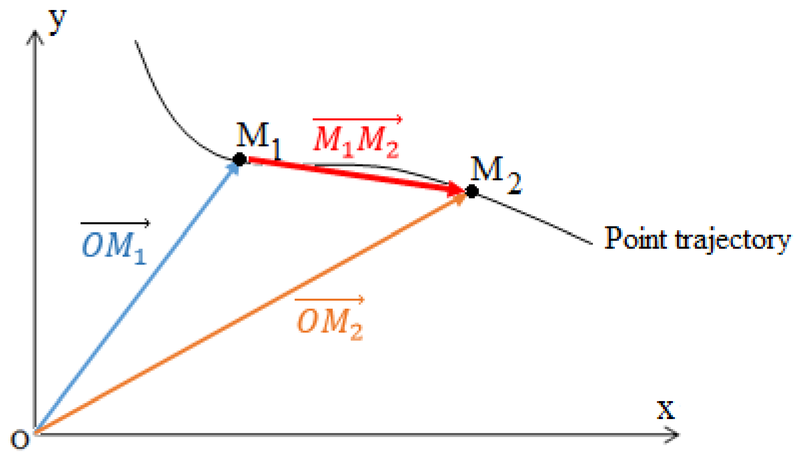
An object's **average velocity** during a time interval Δt is its displacement $\Delta\vec{r}$ divided by Δt:

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\overrightarrow{OM}}{\Delta t}$$

Between two points, $M_1(t_1)$ and $M_2(t_2)$, the displacement vector $\overrightarrow{M_1M_2}$ can be written as:

$$\overrightarrow{M_1M_2} = \overrightarrow{M_1O} + \overrightarrow{OM_2} = \overrightarrow{OM_2} - \overrightarrow{OM_1} = \Delta\overrightarrow{OM} = \Delta\vec{r}$$

$$\vec{v}_{av} = \frac{\overrightarrow{OM_2} - \overrightarrow{OM_1}}{\Delta t}$$



Instantaneous velocity is the limit of the average velocity as Δt approaches to zero. The instantaneous velocity is denoted as vector $\vec{v}$.

$$\vec{v}(t) = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\overrightarrow{OM}}{\Delta t}$$

Δt approaches to zero, it means that the two points are very close.

This expression, in mathematics, represents the definition of the derivative of a function; we can write: $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d\overrightarrow{OM}}{dt}$

$\overrightarrow{OM}$ is the vector position of the material point $M(x, y, z)$: $\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$, So:

$$\boxed{\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}}$$

Instantaneous velocity (velocity only in the following of this chapter) is the derivate of vector position $\overrightarrow{OM}$, we can write it as:

$$\vec{v}(t) = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$$

Where $(\frac{dx}{dt})$ is Leibniz's notation and $(\dot{x})$ is Newton's notation:

$v_x(t),\ v_y(t)$ and $v_z(t)$ represent the components of vector $\vec{v}(t)$, so we write: $\begin{cases} v_x(t) = \frac{dx}{dt} = \dot{x} \\ v_y(t) = \frac{dy}{dt} = \dot{y} \\ v_z(t) = \frac{dz}{dt} = \dot{z} \end{cases}$

From the expression $\vec{v}(t) = \frac{d\vec{r}}{dt}$, we can conclude that the vector $\vec{v}(t)$ and the displacement vectors $d\vec{r}$ are parallel. For an infinitesimal displacement (two points very close), the vector displacement will be parallel to the trajectory. Consequently, the vector of instantaneous velocity $\vec{v}$(t) is always tangent to the trajectory. For that reason, at any point in the trajectory, the vector of velocity is always represented by a vector parallel to the trajectory.

The magnitude of velocity $\vec{v}$ represents the speed of the material point: $\|\vec{v}\| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

If the magnitude of velocity is constant, we say that the motion is uniform.

Application

The position vector of a particle is: $\overrightarrow{OM} = 2t^2\vec{\imath} + (2+3t)\vec{\jmath} + 5t\vec{k}$

1/ What is the instantaneous velocity and speed at t = 2.0 s?

2/ What is the average velocity between 1 s and 3 s?

Solution:

1/ The instantaneous velocity is: $\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$

$$\begin{cases} x(t) = 2t^2 \\ y(t) = 2 + 3t \\ z(t) = 5t \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} = 4t \\ \frac{dy}{dt} = 3 \\ \frac{dz}{dt} = 5 \end{cases} \Rightarrow \vec{v}(t) = 4t\vec{\imath} + 3\vec{\jmath} + 5\vec{k}$$

At

t = 2.0 s: $\vec{v}(t=2) = (4.2)\vec{\imath} + 3\vec{\jmath} + 5\vec{k} = 8\vec{\imath} + 3\vec{\jmath} + 5\vec{k}$

The instantaneous velocity at t = 2 s is: $\boxed{\vec{v}(2\,s) = 8\vec{\imath} + 3\vec{\jmath} + 5\vec{k}}$

The speed at t = 2.0 s represents the magnitude of the instantaneous velocity:

$\|\vec{v}(2\,s)\| = \sqrt{8^2 + 3^2 + 5^2} = 9{,}9\ m/s.$

2/ The average velocity between 1 s and 3 s is: $\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\overrightarrow{OM}}{\Delta t} = \frac{\overrightarrow{OM}(3s) - \overrightarrow{OM}(1s)}{3-1}$

$\overrightarrow{OM}(3s) = 2(3)^2\vec{\imath} + \big(2 + 3(3)\big)\vec{\jmath} + 5(3)\vec{k} = 18\vec{\imath} + 11\vec{\jmath} + 15\vec{k}$

$\overrightarrow{OM}(1s) = 2(1)^2\vec{\imath} + \big(2 + 3(1)\big)\vec{\jmath} + 5(1)\vec{k} = 2\vec{\imath} + 5\vec{\jmath} + 5\vec{k}$

$$\vec{v}_{av} = \frac{\left(18\vec{\imath} + 11\vec{\jmath} + 15\vec{k}\right) - \left(2\vec{\imath} + 5\vec{\jmath} + 5\vec{k}\right)}{2} = \frac{16\vec{\imath} + 6\vec{\jmath} + 10\vec{k}}{2} = 8\vec{\imath} + 3\vec{\jmath} + 5\vec{k}$$

The average velocity between 1 s and 3 s is: $\boxed{\vec{v}_{av} = 8\vec{\imath} + 3\vec{\jmath} + 5\vec{k}}$

Acceleration vector

The instantaneous acceleration vector $\vec{a}$ *or* $\vec{\gamma}$, in the Cartesian coordinates system, at any point in time along its trajectory is obtained from the derivative with regard to the time of the velocity vector:

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

We can write also: $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\overrightarrow{OM}}{dt}\right) = \frac{d^2\overrightarrow{OM}}{dt^2}$

The instantaneous acceleration vector (acceleration vector for the following of this chapter) also can be defined by the second derivative of position vector with respect to time.

$$\boxed{\vec{\boldsymbol{a}}(\boldsymbol{t}) = \frac{\boldsymbol{d^2x}}{\boldsymbol{dt^2}}\vec{\boldsymbol{\imath}} + \frac{\boldsymbol{d^2y}}{\boldsymbol{dt^2}}\vec{\boldsymbol{\jmath}} + \frac{\boldsymbol{d^2z}}{\boldsymbol{dt^2}}\vec{\boldsymbol{k}} = \ddot{\boldsymbol{x}}\vec{\boldsymbol{\imath}} + \ddot{\boldsymbol{y}}\vec{\boldsymbol{\jmath}} + \ddot{\boldsymbol{z}}\vec{\boldsymbol{k}}}$$

Application

The equations of motion of a material point M are: $\begin{cases} x(t) = 2t - 2 \\ y(t) = t^2 - 2t + 1 \end{cases}$

1/ Find the path equation then draw this path.

2/ Find the coordinates of the starting point of movement and the point at time t=1s; Indicate these points on the trajectory.

3/ Write expressions of: vector position $\overrightarrow{OM}(t)$, *velocity vector* $\vec{v}(t)$ *and acceleration vector* $\vec{a}(t)$.

4/ Find velocity vector and acceleration vector at time: $t = 0s$, $t = 1s$. *Draw these vectors on the trajectory.*

Solution:

1/ Trajectory equation: we have $t = \frac{1}{2}x^2 + 1$

We replace t in y equation: $y = \frac{1}{4}x^2$

2/ the starting point of movement corresponds $t = 0s$: $\begin{cases} x(t=0) = -2 \\ y(t=0) = 1 \end{cases} \Rightarrow M_0(-2,1)$

$t = 1s$: $\begin{cases} x(t=1) = 0 \\ y(t=1) = 0 \end{cases} \Rightarrow M_1(0,0)$

3/ The vector position is the vector: $\overrightarrow{OM}(t) = x\vec{\imath} + y\vec{\jmath}$

$\overrightarrow{OM} = (2t-2)\vec{\imath} + (t^2 - 2t + 1)\vec{\jmath}$

The velocity vector of material point is the vector: $\vec{v}(t) = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath}$

$$\begin{cases} \frac{dx}{dt} = 2 \\ \frac{dy}{dt} = 2t - 2 \end{cases} \Longrightarrow \vec{v}(t) = 2\vec{\imath} + (2t-2)\vec{\jmath}$$

The acceleration vector of material point is the vector: $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath}$

$$\begin{cases} \frac{d^2x}{dt^2} = 0 \\ \frac{d^2y}{dt^2} = 2 \end{cases} \Longrightarrow \vec{a}(t) = 0\vec{\imath} + 2\vec{\jmath} = 2\vec{\jmath}$$

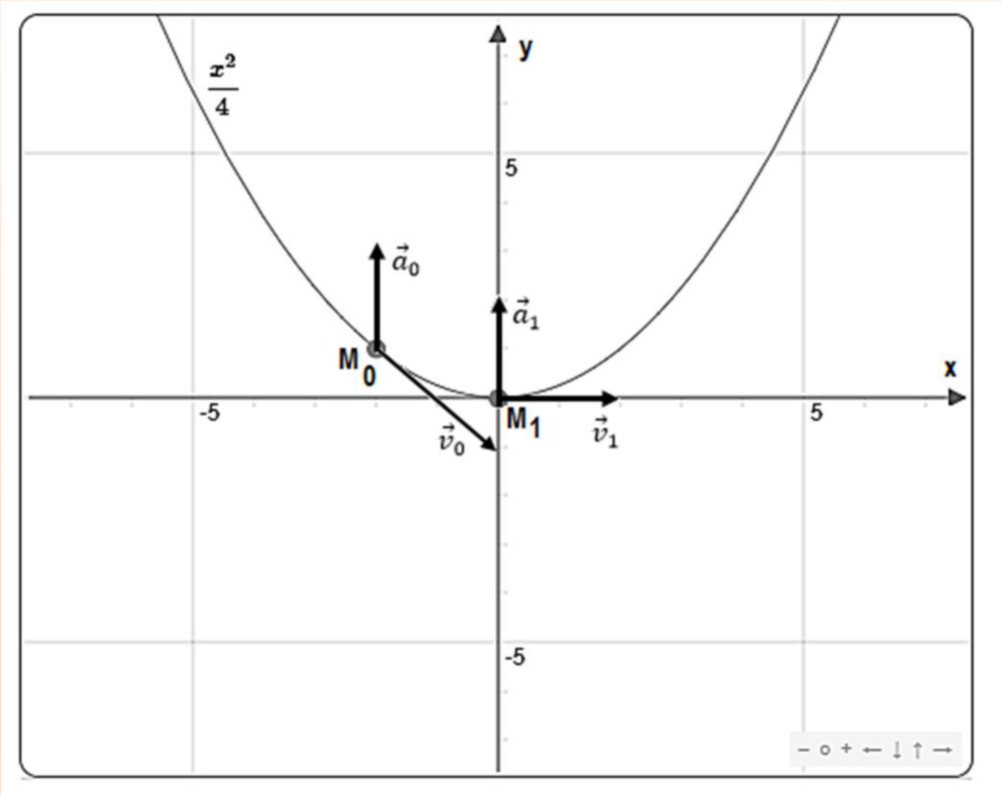
4/ t=0s: $\vec{v}(t=0) = 2\vec{\imath} + (-2)\vec{\jmath}$; $\vec{a}(t=0) = 2\vec{\jmath}$

t=1s: $\vec{v}(t=1) = 2\vec{\imath}$; $\vec{a}(t=1) = 2\vec{\jmath}$

The acceleration vector is always oriented towards the concavity of the trajectory.

The unit of acceleration is $(m.s^{-2} = \frac{m}{s^2})$; an acceleration of 1 m/s^2 means that the speed increases by 1 m/s per second.

The acceleration magnitude is: $\boxed{\|\vec{\boldsymbol{a}}\| = \sqrt{\ddot{\boldsymbol{x}}^2 + \ddot{\boldsymbol{y}}^2 + \ddot{\boldsymbol{z}}^2}}$



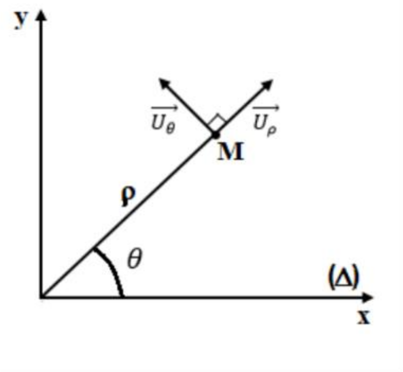
II.2.2 Kinematic quantities in polar coordinates

A material point in the polar coordinates is located with the coordinates (ρ, θ). If the point is in motion, the variables ρ and θ are the functions of time (t);

$\rho = \rho(t)\ ;\ \theta = \theta(t)$

In the case of polar coordinates, the unit vectors $\overrightarrow{U_\rho}$ and $\overrightarrow{U_\theta}$ also vary with time.

The instantaneous position vector $\overrightarrow{OM}$ in polar coordinates is given by: $\overrightarrow{OM} = \rho\overrightarrow{U_\rho}$



The instantaneous velocity vector is: $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(\rho\overrightarrow{U_\rho})}{dt} = \rho.\frac{d\overrightarrow{U_\sigma}}{dt} + \frac{d\rho}{dt}.\overrightarrow{U_\rho}$

$\frac{d\rho}{dt} = \dot{\rho}\,;\frac{d\theta}{dt} = \dot{\theta}$

We have already found that: $\frac{d\overrightarrow{U_\rho}}{dt} = \dot{\theta}.\overrightarrow{U_\theta}$ and $\frac{d\overrightarrow{U_\theta}}{dt} = -\dot{\theta}.\overrightarrow{U_\rho}$

So: $\boldsymbol{\vec{v} = \dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\theta}\overrightarrow{U_\theta}}$ and $\|\boldsymbol{\vec{v}}\| = \sqrt{\boldsymbol{\dot{\rho}^2 + \rho^2\dot{\theta}^2}}$

$\vec{v} = v_\rho\overrightarrow{U_\rho} + v_\theta\overrightarrow{U_\theta}$; $v_\rho = \dot{\rho}$ is the radial component and $v_\theta = \rho\dot{\theta}$ is the orthoradial component.

The instantaneous acceleration vector $\vec{a}$ of point *M* is:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d}{dt}\left(\dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\theta}\overrightarrow{U_\theta}\right) = \frac{d\dot{\rho}}{dt}\overrightarrow{U_\rho} + \rho.\frac{d\overrightarrow{U_\rho}}{dt} + \dot{\rho}\frac{d\dot{\theta}}{dt}\overrightarrow{U_\theta} + \rho\dot{\theta}\frac{d\overrightarrow{U_\theta}}{dt} + \dot{\rho}\dot{\theta}\overrightarrow{U_\theta}$$

$$\vec{a} = \ddot{\rho}\overrightarrow{U_\rho} + \dot{\rho}\dot{\theta}\overrightarrow{U_\theta} + \dot{\rho}\dot{\theta}\overrightarrow{U_\theta} + \rho\ddot{\theta}\overrightarrow{U_\theta} - \rho\dot{\theta}^2\overrightarrow{U_\rho}$$

So: $\boldsymbol{\vec{a}} = (\boldsymbol{\ddot{\rho} - \rho\dot{\theta}^2})\boldsymbol{\overrightarrow{U_\rho}} + (\boldsymbol{2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}})\boldsymbol{\overrightarrow{U_\theta}}$ and $\|\boldsymbol{\vec{a}}\| = \sqrt{(\boldsymbol{\ddot{\rho} - \rho\dot{\theta}^2})^2 + (\boldsymbol{2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}})^2}$

Application

A material point moves in the plane attributed to the polar coordinates $(O, \overrightarrow{U_\rho}, \overrightarrow{U_\theta})$ according to the motion equations:

$\rho = a\theta$ and $\theta = \omega t$

a and ω are constants.

1/ Plot the trajectory of point M in the interval $0 \leq \theta \leq 2\pi$; for a=1.

2/ Find vectors expression of: position $\overrightarrow{OM}$, speed $\vec{v}$, and acceleration $\vec{a}$ in the basis of the polar coordinates $(\overrightarrow{U_\rho}, \overrightarrow{U_\theta})$, conclude their modules.

3/ indicate on the trajectory for the angle $(\theta = \frac{7\pi}{4})$ the base $(\overrightarrow{U_\rho}, \overrightarrow{U_\theta})$, $\overrightarrow{OM}$, $\vec{v}$ and $\vec{a}$.

Solution:

1/ For a=1 and for the famous angles, we calculate ρ:

θ	**0**	**π/6**	**π/4**	**π/3**	**π/2**	**2π/3**	**3π/4**	**5π/6**	**π**	**7π/6**	**5π/4**	**8π/6**	**3π/2**	**10π/6**	**7π/4**	**11π/6**	**2π**
ρ	**0**	**0.52**	**0.78**	**1.04**	**1.57**	**2.09**	**2.35**	**2.61**	**3.14**	**3.66**	**3.92**	**4.18**	**4.71**	**5.23**	**5.49**	**5.75**	**6.28**

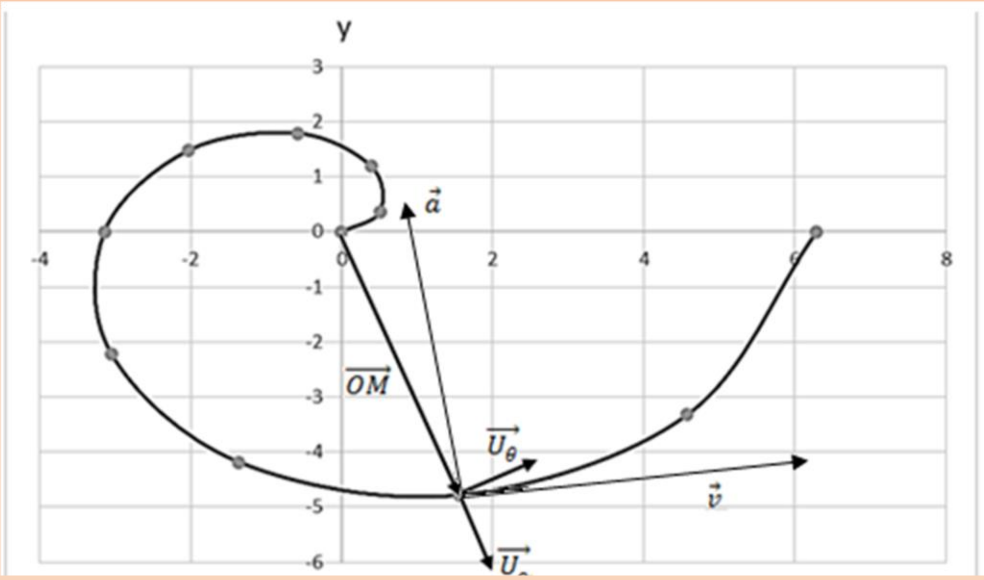
To plot the trajectory, we must divide the Oxy plane into angles π (from 0 to 2π) ;then, for each angle use a unit of measurement, and we indicate the value of ρ (which is the module of $\overrightarrow{OM}$ according to the value in the table. The trajectory is spiral.

2/ $\overrightarrow{OM} = \rho\overrightarrow{U_\rho} = a\theta\overrightarrow{U_\rho} = a\omega t\,\overrightarrow{U_\rho}$; $\|\overrightarrow{OM}\| = a\theta$

$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}(a\theta\overrightarrow{U_\rho}) = a\dot{\theta}\overrightarrow{U_\rho} + a\theta\dot{\theta}\overrightarrow{U_\theta}$; $\theta = \omega t$; $\dot{\theta} = \frac{d\theta}{dt} = \omega$; $\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\omega}{dt} = 0$

So $\vec{v} = a\omega\overrightarrow{U_\rho} + a\omega t\omega\overrightarrow{U_\theta}$;$\vec{v} = a\omega(\overrightarrow{U_\rho} + \omega t\overrightarrow{U_\theta})$; $\|\vec{v}\| = \sqrt{a\dot{\theta}^2 + a\theta\dot{\theta}^2} = a\omega\sqrt{1+\theta^2}$

$\vec{a} = \frac{d\vec{v}}{dt} = 2a\omega^2\overrightarrow{U_\theta} - \mathrm{at}\omega^3\overrightarrow{U_\rho} = a\omega^2(2\overrightarrow{U_\theta} - \theta\overrightarrow{U_\rho})$; $\|\vec{a}\| = a\omega^2\sqrt{4+\theta^2}$



II.2.3 Kinematic quantities in cylindrical coordinates

The position vector in the cylindrical coordinates of a material point M is the same in the polar coordinates; we should just add the component following Oz:

$$\boldsymbol{\overrightarrow{OM} = \rho\overrightarrow{U_\rho} + z\vec{k}}$$

ρ(t), θ(t), and z(t) are the equations of motion.

The magnitude of $\overrightarrow{OM}$ is: $\|\overrightarrow{OM}\| = \sqrt{\rho^2 + z^2}$

The instantaneous velocity vector $\vec{v}$ is: $\boldsymbol{\vec{v} = \dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\theta}\overrightarrow{U_\theta} + \dot{z}\vec{k}}$

$\|\vec{v}\| = \sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2}$

The instantaneous acceleration vector $\vec{a}$ is: $\boldsymbol{\vec{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\overrightarrow{U_\rho} + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right)\overrightarrow{U_\theta} + \ddot{z}\vec{k}}$

$\|\vec{a}\| = \sqrt{(\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})^2 + \ddot{z}\vec{k}}$

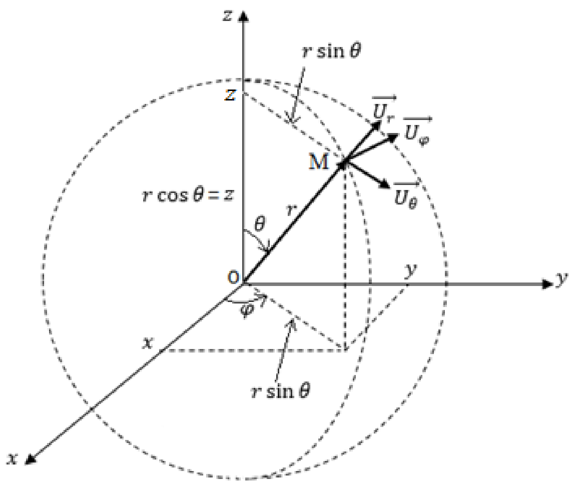
II.2.4 Kinematic quantities in spherical coordinates

The position vector of a material point M in the basic spherical coordinates $\left(\overrightarrow{U_r}, \overrightarrow{U_\theta}, \overrightarrow{U_\varphi}\right)$:

$$\boldsymbol{\overrightarrow{OM} = r\overrightarrow{U_r}}$$

The magnitude of $\overrightarrow{OM}$ is: $\|\overrightarrow{OM}\| = r$

r(t), θ(t), and φ(t) are the equations of motion.



The instantaneous velocity vector of point M is : $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dr}{dt}\overrightarrow{U_r} + r\frac{d\overrightarrow{U_r}}{dt}$

We have seen in the first chapter that the derivatives of the unit vectors $(\overrightarrow{U_r}, \overrightarrow{U_\theta}, \overrightarrow{U_\varphi})$ are:

- $\frac{d\overrightarrow{U_r}}{dt} = \dot{\theta}\overrightarrow{U_\theta} + \dot{\varphi}.\sin\theta\,\overrightarrow{U_\varphi}$
- $\frac{d\overrightarrow{U_\theta}}{dt} = -\dot{\theta}\overrightarrow{U_r} + \dot{\varphi}.\cos\theta\,\overrightarrow{U_\varphi}$
- $\frac{d\overrightarrow{U_\varphi}}{dt} = -\,\dot{\varphi}(\sin\theta\,\overrightarrow{U_r} + \cos\theta\,\overrightarrow{U_\theta}$

So the vector $\vec{v}$ becomes: $\vec{v} = \dot{r}\overrightarrow{U_r} + r\left(\dot{\theta}\overrightarrow{U_\theta} + \dot{\varphi}\sin\theta\,\overrightarrow{U_\varphi}\right)$;

$\Rightarrow\ \boldsymbol{\vec{v} = \dot{r}\overrightarrow{U_r} + r\dot{\theta}\overrightarrow{U_\theta} + r\dot{\varphi}\sin\theta\,\overrightarrow{U_\varphi}}$

The instantaneous acceleration vector $\vec{a}$ is:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\overrightarrow{U_r} + r\dot{\theta}\overrightarrow{U_\theta} + r\dot{\varphi}\sin\theta\,\overrightarrow{U_\varphi}\right)$$

$$\boldsymbol{\vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta\right)\overrightarrow{U_r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta\right)\overrightarrow{U_\theta} + (2\dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\overrightarrow{U_\varphi}}$$

II.2.5 Curvilinear or intrinsic coordinates

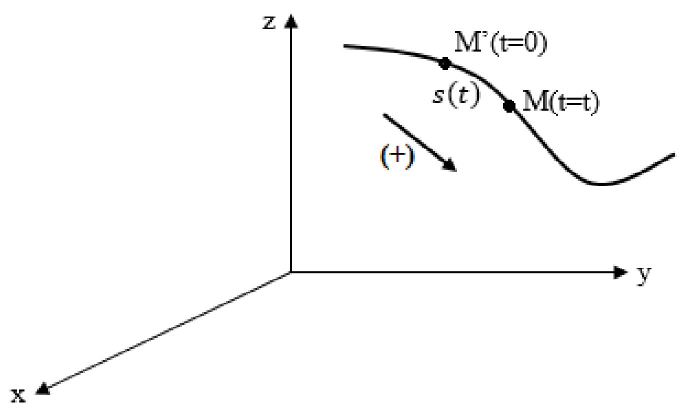
If the trajectory of a material point M is curvilinear, we can use the curvilinear coordinates which are defined as follows:

- Orient the path in the direction of motion.

- Choose a fixed point on the trajectory as the origin of the abscissa.

- We define the curvilinear abscissa $s(t)$ by the length of the arc *MM₀*: $\boxed{\boldsymbol{s(t) = \widehat{MM_0}}}$

The arc length of the path followed by the particle represents the curvilinear abscissa $s(t)$.



Between the two points of the path, the arc length of the path (s) is calculated from the vector position; in the Cartesian Coordinates system $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ and the velocity vector is $\vec{v} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$; $\|\vec{v}\| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$; the arc length of the path (s) between times t_1 and t_2 is:

$$\boldsymbol{s = \int_{t1}^{t2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt}$$

In other words, the arc length of the path equals the integral of speed between two times t_1 and t_2.

Example

Find the arc length of the path, between $t = 1s$ and $t = 5s$, of a particle in motion with position vector: $\vec{r} = (3t-1)\vec{\imath} + (4t+3)\vec{\jmath}$.

Solution:

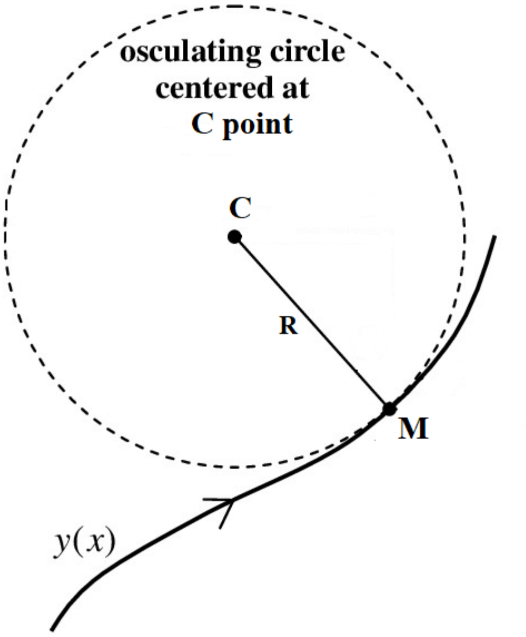
$\vec{v} = 3\vec{\imath} + 4\vec{\jmath}\ ;\ \|\vec{v}\| = \sqrt{3^2 + 4^2} = 5;\ s = \int_{t1}^{t2} 5\, dt = 5t\ |_1^5 = 5(5-1) = 5.4 = 20$

The curvilinear coordinates are characterized by a direct basis $(\overrightarrow{\boldsymbol{U_T}}, \overrightarrow{\boldsymbol{U_N}})$; this basis is orthonormal. It is also called the **Freinet-Serret basis** or the **Intrinsic basis**.

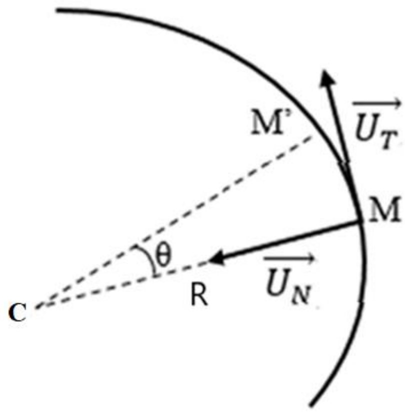
$\overrightarrow{U_T}$ is the tangential unit vector. It is always oriented in the positive direction of movement.

$\overrightarrow{U_N}$ is a unit vector perpendicular on $\overrightarrow{U_T}$; it is always oriented towards the concavity of the trajectory.

Because the path is curved at the point M of it. A circle with center C and radius R ($\boldsymbol{R} = \|\overrightarrow{\boldsymbol{CM}}\|$) is formed; it is called the circle of oscillation. It is tangent to the point M of the path. The radius R of this circle corresponds to the radius of curvature of the path at the point in question, and O is the center of curvature.



The instantaneous velocity is expressed by its module:



$$\vec{v}(M) = \lim_{\Delta t \to 0} \frac{\overrightarrow{MM'}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{dt} . \frac{ds}{ds} = \frac{ds}{dt} . \frac{d\vec{r}}{ds}$$
$$= \frac{ds}{dt} . \overrightarrow{U_T}$$

With: $\frac{d\vec{r}}{ds} = \overrightarrow{U_T}$

$\vec{r}$ is the position vector.

$$\boxed{\vec{\boldsymbol{v}}(\boldsymbol{M}) = \frac{\boldsymbol{ds(t)}}{\boldsymbol{dt}} . \overrightarrow{\boldsymbol{U_T}}}$$

$$\boxed{\|\vec{\boldsymbol{v}}\| = \boldsymbol{v(t)} = \frac{\boldsymbol{ds}}{\boldsymbol{dt}}}$$

The velocity vector is written in the intrinsic basis as: $\vec{\boldsymbol{v}} = \boldsymbol{v(t)}\overrightarrow{\boldsymbol{U_T}} = \frac{\boldsymbol{ds}}{\boldsymbol{dt}}\overrightarrow{\boldsymbol{U_T}} = \dot{\boldsymbol{s}}\overrightarrow{\boldsymbol{U_T}}$

$\vec{v}$ is always tangent to the trajectory.

The instantaneous acceleration vector: $\vec{\boldsymbol{a}}(\boldsymbol{t}) = \frac{\boldsymbol{d\vec{v}}}{\boldsymbol{dt}} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\boldsymbol{v}\overrightarrow{\boldsymbol{U_T}}\right) = \frac{\boldsymbol{dv}}{\boldsymbol{dt}}\overrightarrow{\boldsymbol{U_T}} + \boldsymbol{v(t)}\frac{\boldsymbol{d\overrightarrow{U_T}}}{\boldsymbol{dt}}$

Note:

We have seen that the derivation of unit vectors with regard to the angle θ in the polar coordinate system is the following: $\frac{d\vec{U}_\rho}{d\theta} = \vec{U}_\theta$ and $\frac{d\vec{U}_\theta}{d\theta} = -\vec{U}_\rho$, the derivatives with regard to the time are: $\frac{d\vec{U}_\rho}{dt} = \dot{\theta}\vec{U}_\theta$ and $\frac{d\vec{U}_\theta}{dt} = \dot{\theta} - \vec{U}_\rho$

Accordingly, the rule for differentiating a unit vector with respect to the angle of a rotating vector of constant magnitude is:

The derivative with respect to the angle θ of a unit vector is a unit vector that is directly perpendicular to it (rotation of $\frac{\pi}{2}$ in counterclockwise direction).

So: $\frac{d\vec{U}_T}{d\theta} = \vec{U}_N; \frac{d\vec{U}_N}{d\theta} = -\vec{U}_T$

For differentiating in regard to the time; the result is a unit vector directly perpendicular to it multiplying by the angular speed $\dot{\theta} = \omega$

So: $\frac{d\vec{U}_T}{dt} = \dot{\theta}\,\vec{U}_N$; $\frac{d\vec{U}_N}{dt} = -\dot{\theta}\,\vec{U}_T$

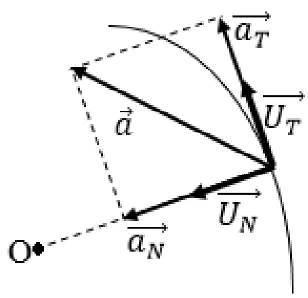
The relationship between linear speed $v(t)$ and angular speed is: $\omega(t) = \dot{\theta}(t)$: $\boxed{\boldsymbol{v = R\omega = R\dot{\theta}}}$

So: $\vec{a}(t) = \frac{dv}{dt}\overrightarrow{U_T} + v\frac{d\overrightarrow{U_T}}{dt} = \frac{dv}{dt}\overrightarrow{U_T} + v\dot{\theta}\overrightarrow{U_N} = \frac{dv}{dt}\overrightarrow{U_T} + v.\frac{v}{R}\overrightarrow{U_N}$

$$\boxed{\vec{\boldsymbol{a}}(\boldsymbol{t}) = \frac{\boldsymbol{dv}}{\boldsymbol{dt}}\overrightarrow{\boldsymbol{U_T}} + \frac{\boldsymbol{v^2}}{\boldsymbol{R}}\overrightarrow{\boldsymbol{U_N}}}$$

We note that the acceleration vector have two components: $\boxed{\boldsymbol{a_T} = \frac{\boldsymbol{dv}}{\boldsymbol{dt}}}$ tangential component and $\boxed{\boldsymbol{a_N} = \frac{\boldsymbol{v^2}}{\boldsymbol{R}}}$ normal component;

$\vec{\boldsymbol{a}}(\boldsymbol{t}) = \boldsymbol{a_T}\overrightarrow{\boldsymbol{U_T}} + \boldsymbol{a_N}\overrightarrow{\boldsymbol{U_N}}$



The magnitude of acceleration is: $\boxed{\|\vec{\boldsymbol{a}}\| = \sqrt{\boldsymbol{a_T}^2 + \boldsymbol{a_N}^2}}$

We can calculate the radius of curvature R by calculating $a_T = \frac{dv}{dt}$ and $a_N = \sqrt{a^2 - a_T{}^2}$.

So: $\boxed{\boldsymbol{R} = \frac{\boldsymbol{v^2}}{\sqrt{\boldsymbol{a^2} - \boldsymbol{a_T}^2}}}$

Application 1

A moving body following the trajectory (C) with the position vector:
$\vec{r}(t) = 3\cos 2t\ \vec{\imath} + 3\sin 2t\ \vec{\jmath} + (8t-4)\vec{k}$ (SI Units)
1/ Find the curvilinear abscissa s between time $t = 1\,s$ and $t = 4\,s$.
2/ Find the unit vector $\overrightarrow{U_T}(t)$, then write it at the Cartesian Coordinate system.
Solution:
1/ We have: $\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = -6\sin 2t\,\vec{\imath} + 6\cos 2t\,\vec{\jmath} + 8\vec{k}$
So $v = \|\vec{v}\| = \sqrt{100} = 10$
$v(t) = \frac{ds}{dt} \Rightarrow ds = vdt \Rightarrow \int_0^s ds = \int_1^4 vdt \Rightarrow s = \int_1^4 10dt = 10t|_1^4 = 10(4-1)$
$s = 30\,m$
2/ The speed vector is always tangential to the trajectory $\overrightarrow{U_T}//\vec{v}$: $\vec{v} = v.\overrightarrow{U_T}$

$$\overrightarrow{U_T} = \frac{\vec{v}}{v} = -\frac{-6\sin 2t\,\vec{\imath} + 6\cos 2t\,\vec{\jmath} + 8\vec{k}}{10}$$

$\overrightarrow{U_T} = -\frac{3}{5}\sin 2t\ \vec{\imath} + \frac{3}{5}\cos 2t\,\vec{\jmath} + \frac{4}{5}\vec{k}$.

Application 2

A particle moves on a curved path. The curvilinear abscissa of this particle is given by:
$s(t) = 3t^2 + 2t + 2$
1/ Determine, as a function of time, the distance traveled by the particle during a time $[0, t[$.
2/ Determine the time needed by M to travel:
The first 50 meters.
The second 50 meters.
3/ Write the expression of speed of M and then find the magnitude of acceleration at time $t = 2s$ knowing that radius of curvature at $t = 2s$ is $10\ m$.
Solution:
1/ The distance traveled by M between 0 to t time is: $\Delta s = s(t) - s(0)$
$s(t) = 3t^2 + 2t + 2$; $s(0) = 2 \Rightarrow \Delta s = 3t^2 + 2t + 2 - 2 \Rightarrow \quad \Delta s = \mathbf{3t^2 + 2t}$
2/ Time needed by M to travel 50m: means $\Delta s = 50\ m$
$\mathbf{3t^2 + 2t = 50} \quad \Rightarrow \quad \mathbf{3t^2 + 2t - 50 = 0}$
This equation has two solutions: $\boldsymbol{t_1 = 3{,}76\ s}$ and: $t_2 = -4{,}42\ s$ (refused because negative sign)

II.3 Different types of motion

Motion is a fundamental concept in physics it is defined as the change in position of an object over time. Motions can be classified into many different types based on the type of path the moving objects take we mention: linear motion, circular motion, periodic motion and random motion. Motion that has no specific pattern.

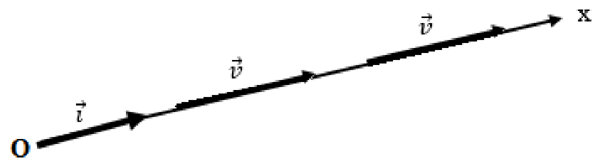
II.3.1 Rectilinear motion

Rectilinear or linear motion is when a body moves in a straight line. However, there are two types of uniform and uniformly varied rectilinear motion in addition to sinusoidal movement. Rectilinear motion requires only one coordinate axis and time to describe a particle's motion.

II.3.1.1 Uniform linear motion

A motion is uniform rectilinear if its trajectory is a straight line and the modulus of the velocity vector is constant; consequently, the acceleration is zero: $\vec{v} = v_0\vec{\imath} = cste$

v_0 is an algebraic quantity (i.e. can be positive or negative)



Equation of motion

We assume that at $t = 0\ s$, the abscissa $x = x_0$.

$$v = v_0 = \frac{dx}{dt} \Rightarrow dx = v_0 dt$$

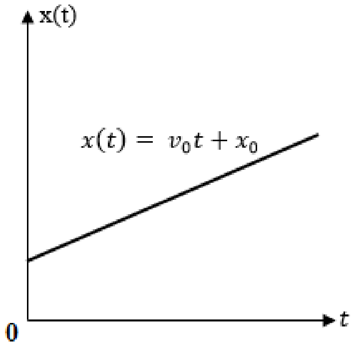
$$\int_{x_0}^{x} dx = \int_0^t v_0 dt \quad \Rightarrow \quad x|_{x_0}^{x} = x - x_0 = v_0 t|_0^t = v_0(t-0)$$

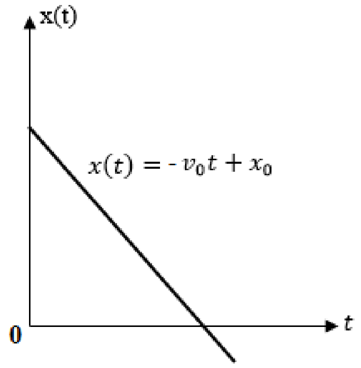
$\boxed{x(t) = v_0 t + x_0}$ this is the equation of motion, x_0 and v_0 depend on the initial conditions.

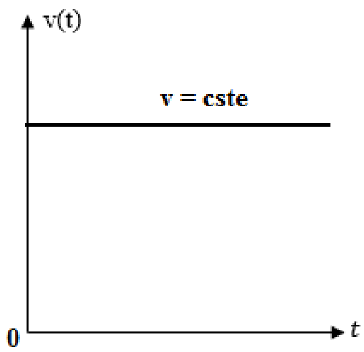
Motion diagrams:

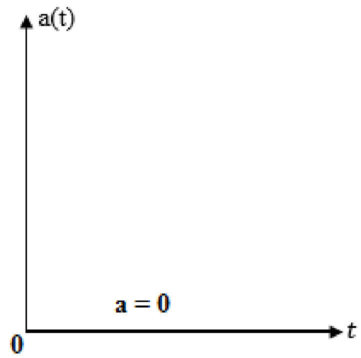
Graphical representations of x(t), v(t), and a(t) of uniform rectilinear motion are shown in the figures below. $\boxed{x(t) = v_0 t + x_0}$

The slope of a distance-time plot gives the velocity v_0 of motion.









II.3.1.2 Uniformly varied rectilinear movement

The movement is rectilinear uniformly varied when the trajectory is straight and the acceleration is constant: $\vec{a} = a_0 \vec{\imath} = cste$

a_0 is an algebraic quantity (positive or negative).

We assume that at $t = 0\ s$; $x = x_0$; $v = v_0$.

$$a = \frac{dv}{dt} = a_0 \Rightarrow dv = a\,dt \Rightarrow \int_{v_0}^{v} dv = \int_0^t a\,dt \Rightarrow v|_{v_0}^{v} = at|_0^t$$

$$v - v_0 = a(t-0) = at \Rightarrow \boxed{\boldsymbol{v(t) = at + v_0}}$$

Equation of motion

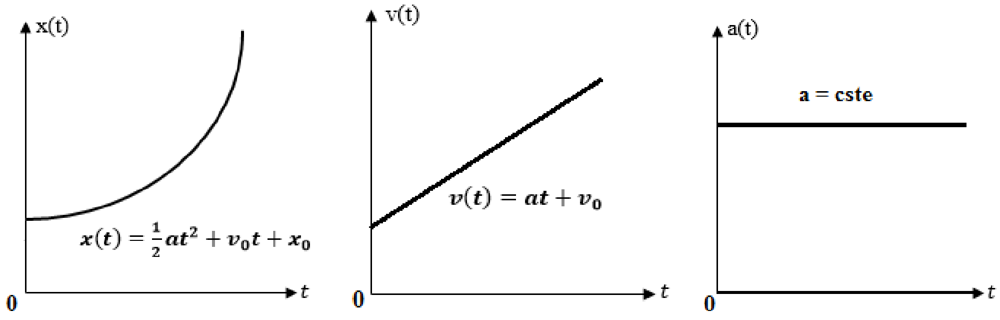
$$v = \frac{dx}{dt} \Rightarrow \quad dx = v\,dt = (at + v_0)dt \Rightarrow \int_{x_0}^{x} dx = \int_0^t (at + v_0)dt$$

$$\Rightarrow \; x - x_0 = \frac{1}{2}at^2 + v_0 t$$

$$\Rightarrow \boxed{\boldsymbol{x(t) = \frac{1}{2}at^2 + v_0 t + x_0}}$$

This is the equation of motion; x_0 and v_0 depend on the initial conditions.

Motion diagrams:



The relation between x and v

We have: $a = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = \frac{dv}{dx}.v \;\Rightarrow\; a\,dx = v\,dv$

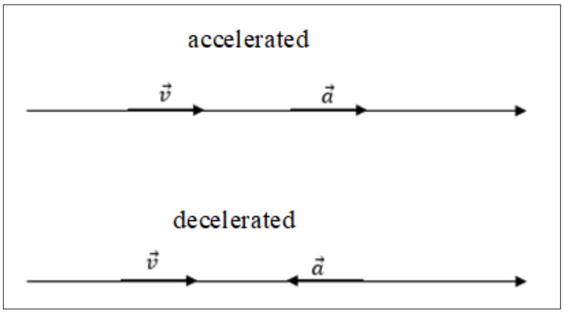
We integrate: $\int_{x_0}^{x} a dx = \int_{v_0}^{v} v dv \;\Rightarrow\; a(x - x_0) = \frac{1}{2}(v^2 - {v_0}^2)$

So: $\boxed{\boldsymbol{v^2 - {v_0}^2 = 2a(x - x_0)}}$

According to the sign of the scalar product $\vec{a}.\vec{v}$ we can know the type of motion accelerated or decelerated (late) movement:

If $\vec{a}.\vec{v} > 0$; $\vec{a}.\vec{v}$ are in the same direction and the movement is accelerated.

If $\vec{a}.\vec{v} < 0$; $\vec{a}.\vec{v}$ are opposed and the movement is decelerated.



Example

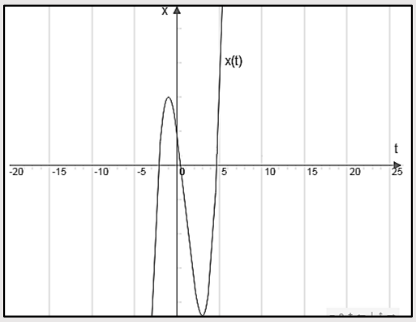
The motion of a body on the axis (Ox) follows the following equations:

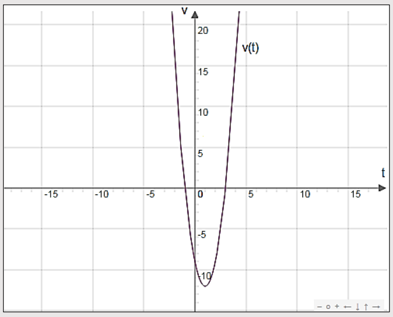
$$x(t) = t^3 - 3t^2 - 9t + 5$$

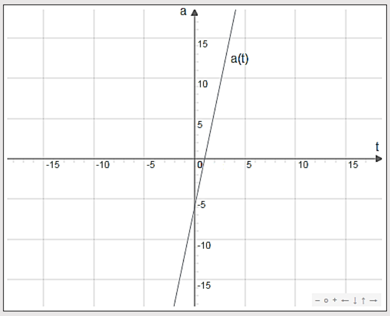
$$v(t) = \frac{dx}{dt} = 3t^2 - 6t - 9$$

$$a(t) = 6t - 6$$

The diagrams of the movement x(t), v(t), and a(t) are shown in the figures below:

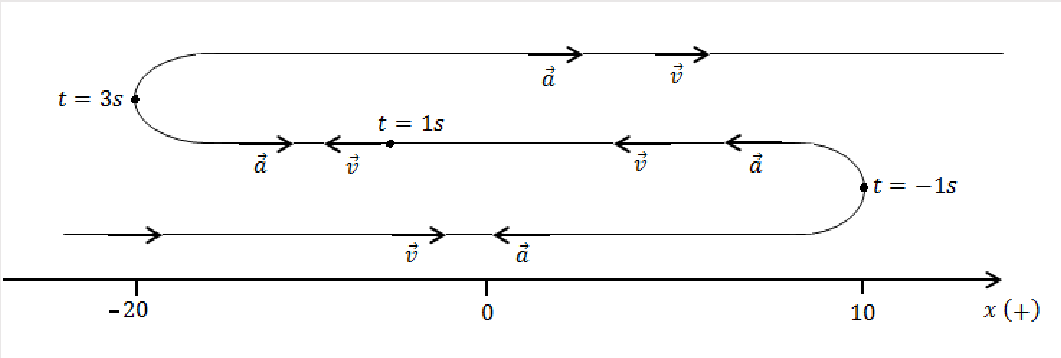






Time domain t	*Acceleration a*	*velocity v*	*Sens of motion*	$v.a$	*Type of motion*
t < -1	*a< 0*	*v > 0*	*(+) positive*	*< 0*	*deceleration*
-1 < t <1	*a <0*	*v <0*	*(-) negative*	*> 0*	*acceleration*
1 < t < 3	*a >0*	*v <0*	*(-) negative*	*< 0*	*deceleration*
t >3	*a >0*	*v > 0*	*(+) positive*	*> 0*	*acceleration*

The following diagram shows the different directions of the velocity and acceleration vectors.



II.3.1.3 Oscillatory motion

Sinusoidal rectilinear motion

Consider a point M moving on Ox axis, its position vector is $\overrightarrow{OM} = x(t)\vec{\imath}$.

A sinusoidal rectilinear motion is a motion that repeats itself in time along a straight line in a sinusoidal pattern, it is periodic.

We say that the motion is rectilinear sinusoidal if its abscissa $x(t)$ has the expression of the form:

$$\boldsymbol{x(t) = X_m \cos(\omega t + \varphi)}$$

When X_m is the maximum amplitude or elongation (in meters).

ω is the pulsation (angular frequency) of the movement in (rad/s)

φ the initial phase, its unit is the radiant (rad)

$(\omega t + \varphi)$ is the instantaneous phase, in radian (rad)

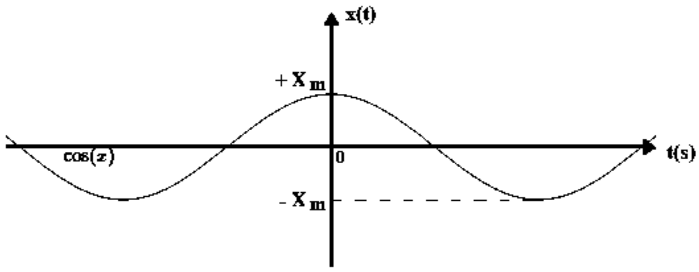
$x(t)$ is the instantaneous elongation or abscissa; it varies between $+X_m$ and $-X_m$

$$-1 \leq \cos(\omega t + \varphi) \leq +1$$

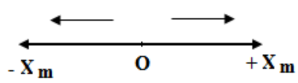
$$-X_m \leq X_m \cos(\omega t + \varphi) \leq +X_m$$

$$\boldsymbol{-X_m \leq \mathrm{x(t)} \leq +X_m}$$

$x(t)$ is the equation of motion. It has a sinusoidal appearance.



The trajectory is therefore a straight-line segment of length $+2X_m$.



The cosine function is a periodic function with a period of 2π. It reproduces after a period; we have the relations:

$$\omega = \frac{2\pi}{T} \Leftrightarrow T = \frac{2\pi}{\omega} \text{ and } T = \frac{1}{f}$$

Period (T) is the time that it takes for one complete oscillation of an object to occur. It is measured in seconds (s).

Frequency (*f*) is the number of oscillations that an object completes per unit of time. It is measured in hertz (Hz), where 1 Hz is equal to 1 oscillation per second.

Angular frequency (ω) is the rate at which an object rotates or oscillates about its equilibrium position. It is measured in radians per second (rad/s).

Sinusoidal instantaneous velocity

$$v = \frac{dx}{dt} = \frac{d}{dt}(X_m \cos(\omega t + \varphi)) = -\omega X_m \sin(\omega t + \varphi)$$

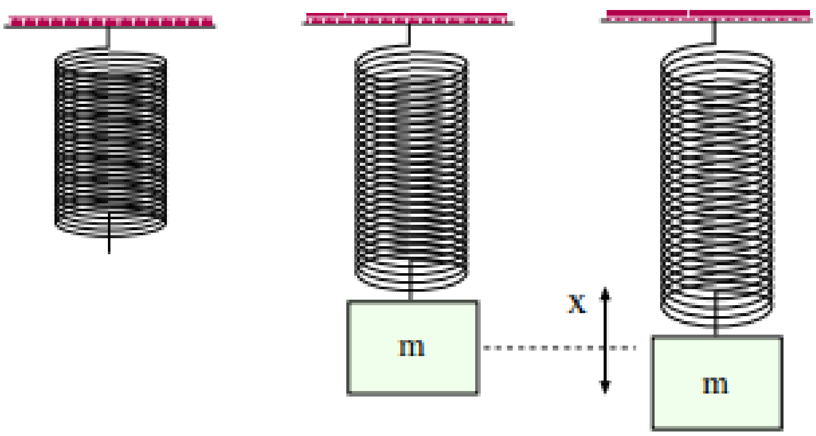
Sinusoidal instantaneous acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}(-\omega X_m \sin(\omega t + \varphi)) = -\omega^2 X_m \cos(\omega t + \varphi) = -\omega^2 x(t)$$

We have: $a = -\omega^2 x(t) \Leftrightarrow \ddot{x} = -\omega^2 x(t) \Rightarrow \boxed{\ddot{\boldsymbol{x}} + \boldsymbol{\omega}^2 \boldsymbol{x} = \mathbf{0}}$

It is a differential equation that has the solution: $\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{X}_{\boldsymbol{m}} \cos(\boldsymbol{\omega t} + \boldsymbol{\varphi})$

The mass attached to a spring system represents an example of sinusoidal rectilinear motion:



II.3.1.4 Circular motion

Circular motion is the motion of an object around a fixed point, following a circular path. It is always carried out in a plane. We can describe this movement in polar or curvilinear coordinates.

Polar coordinates

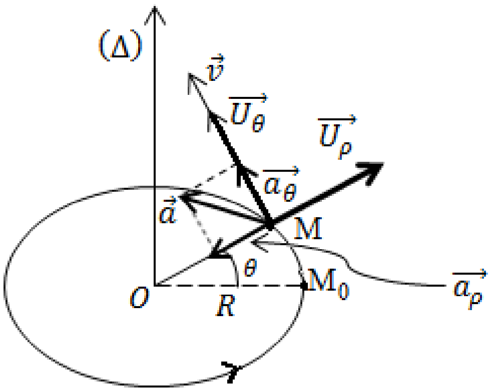
$$\overrightarrow{OM} = R\overrightarrow{U_\rho}$$

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = R\frac{d\overrightarrow{U_\rho}}{dt} = R\dot{\theta}\overrightarrow{U_\theta}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -R\dot{\theta}^2\overrightarrow{U_\rho} + R\ddot{\theta}\,\overrightarrow{U_\theta} = -\frac{v^2}{R}\overrightarrow{U_\rho} + R\ddot{\theta}\,\overrightarrow{U_\theta}$$

Since the relationship between the linear speed v and the angular speed $\dot{\theta}$ is: $v = R\dot{\theta}$

So: $\vec{a} = \overrightarrow{a_\rho} + \overrightarrow{a_\theta}$ with $\overrightarrow{a_\rho} = -\frac{v^2}{R}\overrightarrow{U_\rho}$ and $\overrightarrow{a_\theta} = R\ddot{\theta}\,\overrightarrow{U_\theta}$



Curvilinear coordinates

In curvilinear coordinates, the position of point M in rotation around the center of a circle is defined by s(t) with: $\boxed{s(t) = R.\theta(t)}$

Where R is the radius of the circle and θ is the angle of rotation, varying with time.

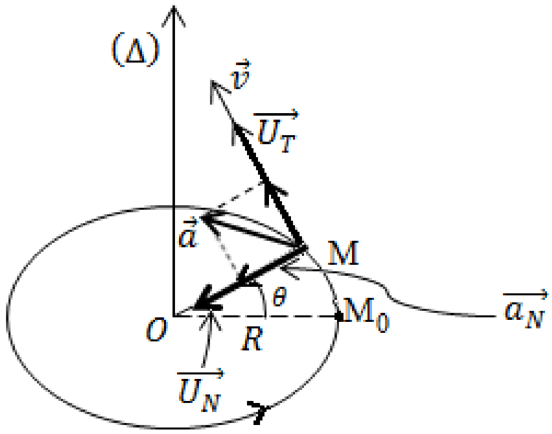
$v(t) = \frac{ds}{dt} \Rightarrow \vec{v} = v\overrightarrow{U_T} = \frac{ds}{dt}\overrightarrow{U_T} = R\dot{\theta}\overrightarrow{U_T}$ so: $\boxed{\vec{v} = v\overrightarrow{U_T}}$; $\boxed{v(t) = R\dot{\theta}}$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\overrightarrow{U_T} + v\frac{d\overrightarrow{U_T}}{dt}$$

$\frac{d\overrightarrow{U_T}}{dt} = \dot{\theta}\overrightarrow{U_N}$; $\frac{dv}{dt} = R\ddot{\theta}$ so: $\vec{a} = R\ddot{\theta}\overrightarrow{U_T} + R\dot{\theta}^2\overrightarrow{U_N}$

$\boxed{\vec{a} = R\ddot{\theta}\overrightarrow{U_T} + \frac{v^2}{R}\overrightarrow{U_N}} \Rightarrow \vec{a} = a_T\overrightarrow{U_T} + a_N\overrightarrow{U_N}$

With: $\boxed{a_T = R\ddot{\theta}}$ et $\boxed{a_N = \frac{v^2}{R} = R\dot{\theta}^2}$



By comparing between the two coordinates, we find that:

$\overrightarrow{U_T} = \overrightarrow{U_\theta}$ and $\overrightarrow{U_N} = -\overrightarrow{U_\rho}$

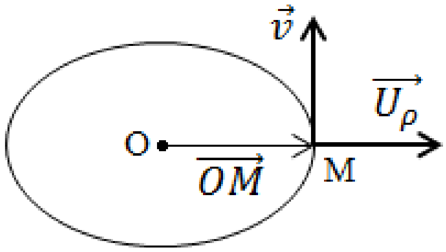
$\|\overrightarrow{U_T}\| = \|\overrightarrow{U_\theta}\| = R\ddot{\theta}$; $\|\overrightarrow{U_N}\| = \|\overrightarrow{U_\rho}\| = \frac{v^2}{R} = R\dot{\theta}^2$.

Angular velocity vector

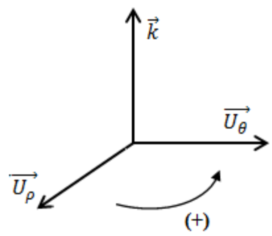
The point *M* rotating around the point *O* has a speed *v* and an angular speed $\dot{\theta} = \omega$.

Now let's find the angular velocity vector $\vec{\omega}$:

We have: $\overrightarrow{OM} = R\overrightarrow{U_\rho}$; $\vec{v} = R\dot{\theta}\overrightarrow{U_\theta}$



We have the direct basis $(\overrightarrow{U_\rho}, \overrightarrow{U_\theta})$ of the polar coordinates, if we add the unit vector $\vec{k}$ to this basis such as $(\overrightarrow{U_\rho}, \overrightarrow{U_\theta}, \vec{k})$ is a direct basis.



So we have: $\begin{cases} \overrightarrow{U_\rho} \wedge \overrightarrow{U_\theta} = \vec{k} \\ \overrightarrow{U_\theta} \wedge \vec{k} = \overrightarrow{U_\rho} \\ \vec{k} \wedge \overrightarrow{U_\rho} = \overrightarrow{U_\theta} \end{cases}$

The relation: $\vec{k} \wedge \overrightarrow{U_\rho} = \overrightarrow{U_\theta}$ we can write $\vec{v} = R\dot{\theta}\overrightarrow{U_\theta} = R\dot{\theta}(\vec{k} \wedge \overrightarrow{U_\rho})$

So: $\vec{v} = \dot{\theta}\vec{k} \wedge R\,\overrightarrow{U_\rho}$

While: $R\,\overrightarrow{U_\rho} = \overrightarrow{OM}$

So: $\vec{v} = \dot{\theta}\vec{k} \wedge \overrightarrow{OM}$

if we assume that $\dot{\theta}\vec{k}$ represents a vector: $\boxed{\vec{\boldsymbol{\omega}} = \dot{\boldsymbol{\theta}}\vec{\boldsymbol{k}}}$, it is the angular velocity vector of magnitude $\|\vec{\omega}\| = \dot{\theta}$

We can write $\boxed{\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \wedge \overrightarrow{\boldsymbol{OM}}}$

The vector $\vec{\omega}$ is oriented towards Oz axis $(\vec{k})$; $\boldsymbol{\vec{\omega} = \omega\vec{k}}$

It is perpendicular to the plane of rotation.

Angular acceleration vector

The linear acceleration vector $\vec{a}$ is given by: $\vec{a} = \frac{d\vec{v}}{dt}$

So: $\vec{a} = \frac{d}{dt}\left(\vec{\omega}\wedge\overrightarrow{OM}\right) = \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{OM} + \vec{\omega}\wedge\frac{d\overrightarrow{OM}}{dt}$

$\vec{\omega} = \dot{\theta}\vec{k} \Rightarrow \frac{d\vec{\omega}}{dt} = \ddot{\theta}\vec{k}$; $\ddot{\theta}$ is the angular acceleration of motion.

$$\boxed{\boldsymbol{\vec{a} = \ddot{\theta}\vec{k}\wedge\overrightarrow{OM} + \vec{\omega}\wedge\vec{v}}}$$

Particular cases

Centripetal motion

If the angular velocity is constant $\dot{\theta} = cte \Rightarrow \ddot{\theta} =$ (the angular acceleration is zero)

$\vec{a} = \vec{\omega}\wedge\vec{v} = \dot{\theta}\vec{k}\wedge \mathrm{R}\dot{\theta}\overrightarrow{U_\theta} = R\,\dot{\theta}^2\vec{k}\wedge\overrightarrow{U_\theta}$

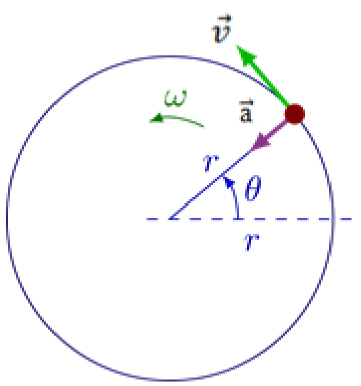
$\vec{k}\wedge\overrightarrow{U_\theta} = -\overrightarrow{U_\rho}$

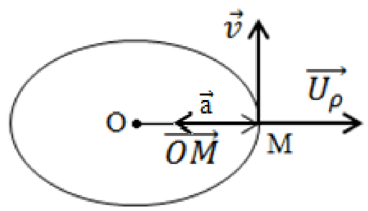
$\Rightarrow \boxed{\boldsymbol{\vec{a} = -R\,\dot{\theta}^2\,\overrightarrow{U_\rho}}}$ we notice that $\vec{a}$ is always parallel to the unit vector $\overrightarrow{U_\rho}$ but in the opposite direction.

$\vec{a} = -\dot{\theta}^2 R\,\overrightarrow{U_\rho} = -\dot{\theta}^2\overrightarrow{OM}$

$$\boxed{\vec{a} = -\dot{\theta}^2\overrightarrow{OM}}$$

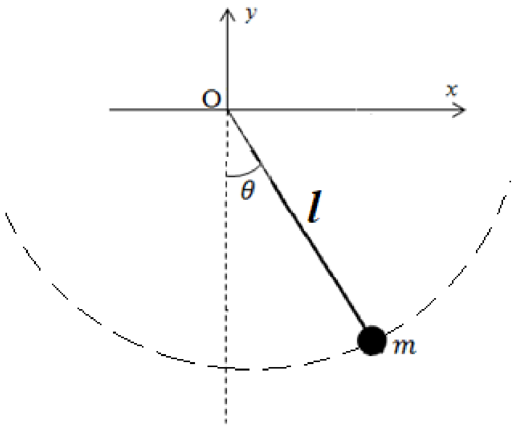
$\vec{a}$ is always parallel to the position vector $\overrightarrow{OM}$ and always perpendicular to the linear velocity $\vec{v}$. We say that the motion is **uniform circular**, and **the acceleration** is **centripetal** (A centripetal motion is a specific type of circular motion in which an object moves in a circular path with a constant speed)

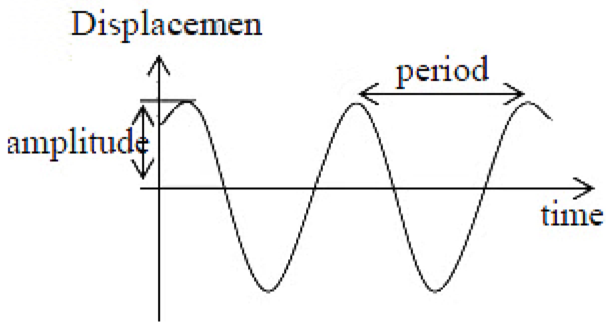




Sinusoidal and circular motion

There is a correlation between sinusoidal motion and uniform circular motion. The oscillating object's position changes sinusoidally with time. Take an example of pendulum, the most basic type of pendulums. It consists of a mass attached to a string of fixed length. When the mass (called bob) is displaced from its equilibrium position and released, it swings back and forth in a regular pattern, over time, the amplitude draws a sine graphics. We say that the pendulum oscillates around the equilibrium position; it sweeps out an arc of a circle of radius *l*.





In this case the equation of motion is described by the angle θ varied with respect to time:

$$\boldsymbol{\theta(t) = \theta_0\, cos(\omega t + \varphi)}$$

Where $\boldsymbol{\theta_0}$ is the initial angular displacement, and $\omega = \sqrt{\frac{l}{g}}$ is the natural frequency of the motion.

The period of this system is $\boldsymbol{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}}$

φ is the initial phase.

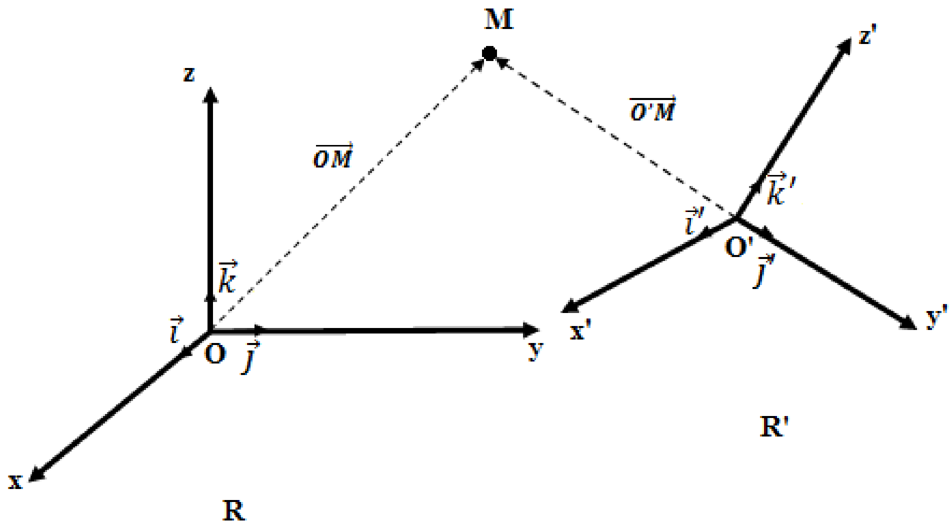
II.3.3 Relative motion

We use the notion of relative movement in the study of the movement of a material point in relation to two references, one is fixed R (O,$\vec{\imath}$,$\vec{\jmath}$, $\vec{k}$) and the other R'(o',$\vec{\imath}'$, $\vec{\jmath}'$, $\vec{k}'$) is mobile relative to R. We call R the absolute reference frame (Ra) and (R'or Rr) the relative reference frame (movable with respect to time in the absolute reference frame (Ra)). The fixed frame serves as a reference for describing the motion of objects in the relative frame.

The material point M is moving in space. It is defined by the coordinates (x, y, z) in the absolute reference frame (R_a) and by (x', y', z') in the movable reference frame (R').

Absolute motion is the motion of M relative to (R_a).

Relative motion is the motion of M relative to (R_r).



The absolute quantities related to the absolute reference R_a:

Position vector $\overrightarrow{OM}$: $\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

absolute velocity vector $\vec{v}_a$: $\vec{v}_a = \frac{d\,\overrightarrow{OM}}{dt}\Big|_{Ra} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$

The basis of the absolute reference ($\vec{\imath}$, $\vec{\jmath}, \vec{k}$) is fixed, so $\frac{d\vec{\imath}}{dt} = \frac{d\vec{\jmath}}{dt} = \frac{d\vec{\jmath}}{dt} = \vec{0}$

Absolute acceleration $\vec{a}_a$: $\vec{a}_a = \frac{d\vec{v}_a}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$

The relative quantities related to the relative reference Rr:

The relative position vector: $\boxed{\overrightarrow{O'M} = x'\vec{\imath}' + y'\vec{\jmath}' + z'\vec{k}'}$

The relative velocity vector $\vec{v}_r$: $\vec{v}_r = \frac{d\overrightarrow{O'M}}{dt}\Big|_R$

($\vec{\imath}'$,$\vec{\jmath}'$,$\vec{k}'$) is a fixed base with respect to R_r: $\frac{d\vec{\imath}'}{dt}\Big|_{R_r} = \frac{d\vec{\jmath}'}{dt}\Big|_{R_r} = \frac{d\vec{k}'}{dt}\Big|_{R_r} = \vec{0}$

$\vec{v}_r = \dot{x}'\vec{\imath}' + \dot{y}'\vec{\jmath}' + \dot{z}'\vec{k}'$

The acceleration relative vector $\vec{a}_r$: $\vec{a}_r = \frac{d^2\overrightarrow{O'M}}{dt^2}\Big|_{R_r} = \ddot{x}'\vec{\imath}' + \ddot{y}'\vec{\jmath}' + \ddot{z}'\vec{k}'$

II.3.3.1 Velocity composition law

Consider two absolute reference frame Ra and relative Rr; Ra is fixed and Rr is in translational movement. The point M is in motion, we define its position vector ($\overrightarrow{OM}$) as:

$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$$

The coordinates of the point M and O' with respect to the R_a and R_r references are:

$$M_{/Ra}(x, y, z) \Rightarrow \overrightarrow{OM} = \mathrm{x}\vec{\imath} + \mathrm{y}\vec{\jmath} + \mathrm{z}\vec{k}$$

$$O'_{/Ra}(x_0, y_0, z_0) \Rightarrow \overrightarrow{OO'} = \mathrm{x}_0\vec{\imath} + \mathrm{y}_0\vec{\jmath} + \mathrm{z}_0\vec{k}$$

$$M_{/Rr}(x', y', z') \Rightarrow \overrightarrow{O'M} = \mathrm{x}'\vec{\imath}' + \mathrm{y}'\vec{\jmath}' + \mathrm{z}'\vec{k}'$$

$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M} = x_0\vec{\imath} + y_0\vec{\jmath} + z_0\vec{k} + x'\vec{\imath}' + y'\vec{\jmath}' + z'\vec{k}'$$

Absolute velocity vector is:

$$\vec{v}_a = \frac{d\overrightarrow{OM}}{dt}\Big|_{R_a} = \frac{d}{dt}\left(\overrightarrow{OO'}\right) + \frac{d}{dt}\left(\overrightarrow{O'M}\right) = \frac{d}{dt}(x_0\vec{\imath} + y_0\vec{\jmath} + z_0\vec{k} + x'\vec{\imath}' + y'\vec{\jmath}' + z'\vec{k}')$$

$$= \frac{dx_0}{dt}\vec{\imath} + \frac{dy_0}{dt}\vec{\jmath} + \frac{dz_0}{dt}\vec{k} + x'\frac{d\vec{\imath}'}{dt} + y'\frac{d\vec{\jmath}'}{dt} + z'\frac{d\vec{k}'}{dt} + \frac{dx'}{dt}\vec{\imath}' + \frac{dy'}{dt}\vec{\jmath}' + \frac{dz'}{dt}\vec{k}'$$

$$\boxed{\vec{v}_a = \frac{dx_0}{dt}\vec{\imath} + \frac{dy_0}{dt}\vec{\jmath} + \frac{dz_0}{dt}\vec{k} + x'\frac{d\vec{\imath}'}{dt} + y'\frac{d\vec{\jmath}'}{dt} + z'\frac{d\vec{k}'}{dt} + \frac{dx'}{dt}\vec{\imath}' + \frac{dy'}{dt}\vec{\jmath}' + \frac{dz'}{dt}\vec{k}'}$$

$(\vec{O}, \vec{\imath}, \vec{\jmath}, \vec{k})$ is fixed with respect to $R_r \Rightarrow \frac{d\vec{\imath}}{dt} = \frac{d\vec{\jmath}}{dt} = \frac{d\vec{k}}{dt} = \vec{o}$

$(\vec{\imath}', \vec{\jmath}', \vec{k}')$ is mobile with respect to Rr $\Rightarrow \frac{d\vec{\imath}'}{dt} \neq \vec{o}, \frac{d\vec{\jmath}'}{dt} \neq \vec{o}, \frac{d\vec{k}_\iota}{dt} \neq \vec{o}$

We have:

$$\vec{v}_e = \frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{\imath}'}{dt} + y'\frac{d\vec{\jmath}'}{dt} + z'\frac{d\vec{k}'}{dt} = \dot{x}_0\vec{\imath} + \dot{y}_0\vec{\jmath} + \dot{z}_0\vec{k} + x'\frac{d\vec{\imath}'}{dt} + y'\frac{d\vec{\jmath}'}{dt} + z'\frac{d\vec{k}'}{dt}$$

$\vec{v}_e$ is the training velocity; it is the speed of the moving frame Rr relative to Ra.

And we have: $\vec{v}_r = \dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'}$

$\vec{v}_r$ is the relative velocity of point M, it is the velocity of M relative to R_r.

So we can write: $\boxed{\vec{\boldsymbol{v}}_{\boldsymbol{a}} = \vec{\boldsymbol{v}}_{\boldsymbol{e}} + \vec{\boldsymbol{v}}_{\boldsymbol{r}}}$ this relation is called the velocity composition law.

Note:

If R_r is fixed $\Rightarrow \vec{v}_e = \vec{0} \Rightarrow \vec{v}_a = \vec{v}_r$

If M is fixed in R_r $\Rightarrow \vec{v}_r = \vec{0} \Rightarrow \vec{v}_a = \vec{v}_e$

II.3.3.2 Acceleration composition law

The absolute acceleration of M with respect to R_a is:

$$\vec{a}_a = \frac{d^2\overrightarrow{OM}}{dt^{\,2}} = \frac{d\vec{v}_a}{dt} = \frac{d(\vec{v}_e + \vec{v}_r)}{dt}$$

$$= \frac{d}{dt}\left(\frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}\right) + \frac{d}{dt}\left(\frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}\right)$$

$$\vec{a}_a = \frac{d^2\overrightarrow{OO'}}{dt^2} + x'\frac{d^2\vec{\imath'}}{dt^2} + y'\frac{d^2\vec{\jmath'}}{dt^2} + z'\frac{d^2\vec{k'}}{dt^2} + \frac{dx'}{dt}\frac{d\vec{\imath'}}{dt} + \frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt} + \frac{d^2x'}{dt^2}\vec{\imath'} + \frac{d^2y'}{dt^2}\vec{\jmath'}$$
$$+ \frac{d^2z'}{dt^2}\vec{k'} + \frac{dx'}{dt}\frac{d\vec{\imath'}}{dt} + \frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}$$

$$\vec{a}_a = \left[\frac{d^2\overrightarrow{OO'}}{dt^2} + x'\frac{d^2\vec{\imath'}}{dt^2} + y'\frac{d^2\vec{\jmath'}}{dt^2} + z'\frac{d^2\vec{k'}}{dt^2}\right] + \left[\frac{d^2x'}{dt^2}\vec{\imath'} + \frac{d^2y'}{dt^2}\vec{\jmath'} + \frac{d^2z'}{dt^2}\vec{k'}\right]$$
$$+ 2\left[\frac{dx'}{dt}\frac{d\vec{\imath'}}{dt} + \frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right]$$

$$\boxed{\vec{\boldsymbol{a}}_{\boldsymbol{a}} = \vec{\boldsymbol{a}}_{\boldsymbol{e}} + \vec{\boldsymbol{a}}_{\boldsymbol{r}} + \vec{\boldsymbol{a}}_{\boldsymbol{c}}}$$

With: $\vec{a}_a$ is the absolute acceleration of M with respect to R_a.

$\vec{a}_e$ is the training acceleration. It is the acceleration of the reference R_r relative to the reference R_a:

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt^2} + x'\frac{d^2\overrightarrow{\imath'}}{dt^2} + y'\frac{d^2\overrightarrow{\jmath'}}{dt^2} + z'\frac{d^2\overrightarrow{k'}}{dt^2}$$

$\vec{a}_r$ is the relative acceleration of M with respect to R_r:

$$\vec{a}_r = \ddot{x}'\overrightarrow{\imath'} + \ddot{y}'\overrightarrow{\jmath'} + \ddot{z}'\overrightarrow{k'}$$

$\vec{a}_c$ is the Coriolis acceleration; it is a complementary acceleration (from its author Gaspard Coriolis 1792-1843 who established it in 1832) (it has no physical meaning).

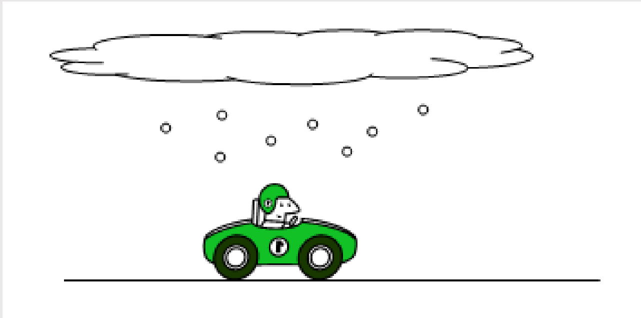
$$\vec{a}_c = 2\left[\dot{x}'\frac{d\overrightarrow{\imath'}}{dt} + \dot{y}'\frac{d\overrightarrow{\jmath'}}{dt} + \dot{z}'\frac{d\overrightarrow{k'}}{dt}\right]$$

In the translation movement from R_r into R_a, the vectors $\overrightarrow{\imath'}, \overrightarrow{\jmath'}, \overrightarrow{k'}$ remain fixed and therefore: $\frac{d\overrightarrow{\imath'}}{dt} = \frac{d\overrightarrow{\jmath'}}{dt} = \frac{d\overrightarrow{k'}}{dt} = \vec{0}$ and so: $\vec{a}_c = \vec{0}$

$\vec{a}_c$ also is canceled if M is fixed with respect to Rr.

Example

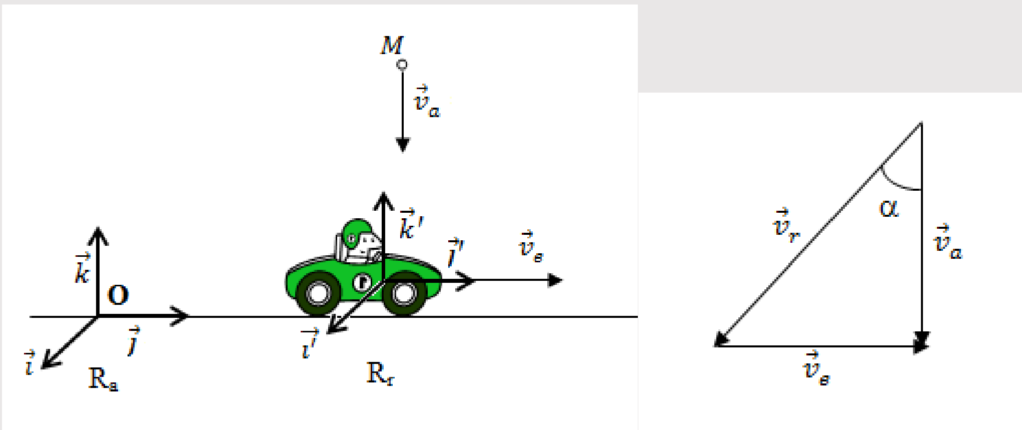
Raindrops fall with a speed equal to 30 m/s. What is the value of speed? do these drops hit the car that is moving at a speed equal to 80 km/h? at what angle do they fall on the car?



Solution:

We must choose a fixed (absolute) reference point, a mobile (relative) reference point, and a material point M that is in motion.

We consider that the earth as an absolute reference, the car as a relative reference, and the drop of water as the point M.



According to the law of composition of speeds: $\vec{v}_a = \vec{v}_e + \vec{v}_r$

$\vec{v}_a$ *the speed of the water drops relative to the earth:* $v_a = 30\ m/s\ ;\ \vec{v}_a = -30\,\vec{k}$

$\vec{v}_e$ *the reference's frame training speed Rr in relation to Ra; this is the speed of the car*

$v_e = 80\ \frac{Km}{h} = 80 \times \frac{10^3}{3600} = 22\ m/s\!:\ \vec{v}_e = 22\,\vec{j}$

It remains just the relative speed v_r *unknown.*

$\vec{v}_a = \vec{v}_e + \vec{v}_r \Rightarrow \vec{v}_r = \vec{v}_a - \vec{v}_e\ \Rightarrow \boxed{\vec{v}_r = -30\,\vec{k} - 22\,\vec{j}}$

We notice that $\vec{v}_r$ *has two components.*

$\|\vec{v}_r\| = \sqrt{22^2 + 30^2} = 37.33\ m/s$

So the speed with which the water drops hit the car is 37.33 m/s

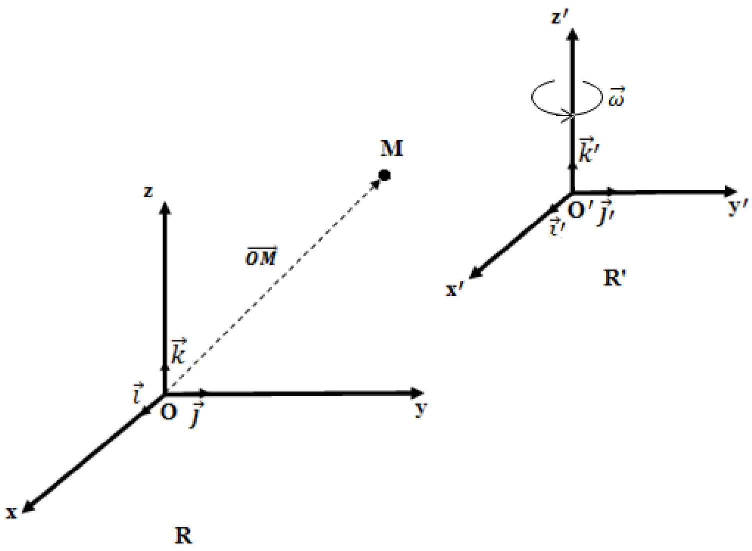
Calculation of the angle: from the triangle, we have: $\tan\alpha\ = \frac{v_e}{v_a} = \frac{22}{30} = 0.73$

$\alpha = arctan\, 0.73 = 36.25°.$

II.3.3.3 Case of rotational movement of the relative reference frame

On the assumption that the axe of rotation reference $\mathbf{R_r}$ around $\mathbf{R_a}$ is the axe Oz.

a- Velocity composition law



We have seen that: $\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$ et $\vec{v}_a = \vec{v}_e + \vec{v}_r$

$$\vec{v}_a = \dot{x}_0\vec{\imath} + \dot{y}_0\vec{\jmath} + \dot{z}_0\vec{k} + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt} + \dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'}$$

To calculate $\frac{d\vec{\imath'}}{dt}, \frac{d\vec{\jmath'}}{dt}$ et $\frac{d\vec{k'}}{dt}$, we assume that the angle of rotation of R_r around R_a is $\vec{\omega}_{Rr/Ra}$(we put $\vec{\omega}_{Rr/Ra} = \vec{\omega}$)

$\vec{\omega} = \dot{\theta}\vec{k} = \dot{\theta}\vec{k'}$ because $\vec{k}$ is parallel to $\vec{k}'$.

$$\frac{d\vec{\imath'}}{dt} = \frac{d\vec{\imath'}}{d\theta}.\frac{d\theta}{dt} = \dot{\theta}\frac{d\vec{\imath'}}{d\theta}$$

We have already seen that: $\frac{d\vec{\imath'}}{d\theta} = \vec{\jmath'}$ and $\frac{d\vec{\jmath'}}{d\theta} = -\vec{\imath'}$ (In a plane of rotation with an angle θ)

$\Rightarrow \frac{d\vec{\imath'}}{dt} = \dot{\theta}\vec{\jmath'}$ and because $(\vec{\imath'}, \vec{\jmath'}, \vec{k'})$ is a direct basis so: $\vec{\imath'}\wedge\vec{\jmath'} = \vec{k'}, \vec{\jmath'}\wedge\vec{k'} = \vec{\imath'}$ and $\vec{k'}\wedge\vec{\imath'} = \vec{\jmath'}$

We replace the third equation $(\vec{k'}\wedge\vec{\imath'} = \vec{\jmath'})$ in the formulate of $\frac{d\vec{\imath'}}{dt}$, we find:

$\frac{d\vec{\imath'}}{dt} = \dot{\theta}.\vec{k'}\wedge\vec{\imath'} \Rightarrow \boxed{\frac{d\vec{\imath'}}{dt} = \vec{\omega}_{Rr/Ra}\wedge\vec{\imath'}}$

The same applies to $\frac{d\vec{\jmath'}}{dt}$ we find: $\frac{d\vec{\jmath'}}{dt} = \frac{d\vec{\jmath'}}{d\theta}.\frac{d\theta}{dt} = \dot{\theta}\frac{d\vec{\jmath'}}{d\theta} = -\dot{\theta}\vec{\imath'}$

Et $\vec{\jmath'}\wedge\vec{k'} = \vec{\imath'} \Rightarrow \frac{d\vec{\jmath'}}{dt} = -\dot{\theta}\vec{\jmath'}\wedge\vec{k'} = -\vec{\jmath'}\wedge\dot{\theta}\vec{k'} = \dot{\theta}\vec{k'}\wedge\vec{\jmath'}$

$$\boxed{\frac{d\vec{\jmath'}}{dt} = \vec{\omega}\wedge\vec{\jmath'}}$$

For $\frac{d\vec{k'}}{dt}$ we find: $\frac{d\vec{k'}}{dt} = \dot{\theta}\frac{d\vec{k'}}{d\theta} = \vec{0}$

We can put the zero vector $\vec{0} = \vec{k'}\wedge\vec{k'}$ (the vector product between two parallel vectors is zero).

$\frac{d\vec{k'}}{dt} = \dot{\theta}.\vec{k'}\wedge\vec{k'} \Rightarrow \boxed{\frac{d\vec{k'}}{dt} = \vec{\omega}\wedge\vec{k'}}$

$$\vec{v}_a = \dot{x}_0\vec{\imath} + \dot{y}_0\vec{\jmath} + \dot{z}_0\vec{k} + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt} + \dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'}$$

$$= \dot{x}_0\vec{\imath} + \dot{y}_0\vec{\jmath} + \dot{z}_0\vec{k} + x'\vec{\omega}\wedge\vec{\imath'} + y'\vec{\omega}\wedge\vec{\jmath'} + z'\vec{\omega}\wedge\vec{k'} + \dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'}$$

$$\Rightarrow \vec{v}_a = \dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'} + \frac{d\overline{OO'}}{dt} + \vec{\omega}\wedge\left(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}\right)$$

$$\Rightarrow \boxed{\vec{v}_a = \dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'} + \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\wedge\overrightarrow{O'M}}$$

So: $\vec{v}_a = \vec{v}_r + \vec{v}_e$ with $\vec{v}_e = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\wedge\overrightarrow{O'M}$

Note:

If R_r is in circular rotation around R_a (O' coinciding with O) $\Rightarrow \frac{d\overrightarrow{OO'}}{dt} = \vec{0}$

$\Rightarrow \vec{v}_e = \vec{\omega}\wedge\overrightarrow{O'M}$ thus $\vec{v}_a = \dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'} + \vec{\omega}\wedge\overrightarrow{O'M}$.

b- Acceleration composition law

The absolute acceleration $\vec{a}_a$ is given by:

$$\vec{a}_a = \frac{d\vec{v}_a}{dt} = \frac{d}{dt}\left(\dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'} + \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\wedge\overrightarrow{O'M}\right)$$

$$= \ddot{x}'\vec{\imath'} + \ddot{y}'\vec{\jmath'} + \ddot{z}'\vec{k'} + \ddot{x}_0\vec{\imath'} + \ddot{y}_0\vec{\jmath'} + \ddot{z}_0\vec{k'} + \dot{x}'\frac{d\vec{\imath'}}{dt} + \dot{y}'\frac{d\vec{\jmath'}}{dt} + \dot{z}'\frac{d\vec{k'}}{dt} + \frac{d}{dt}\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)$$

$$\frac{d}{dt}\left(\vec{\omega}\wedge\overrightarrow{O'M}\right) = \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M} + \vec{\omega}\wedge\frac{d(\overrightarrow{O'M})}{dt}$$

$$\Rightarrow \quad \vec{a}_a = \ddot{x}'\vec{\imath'} + \ddot{y}'\vec{\jmath'} + \ddot{z}'\vec{k'} + \ddot{x}_0\vec{\imath'} + \ddot{y}_0\vec{\jmath'} + \ddot{z}_0\vec{k'} + \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M} + \vec{\omega}\wedge\frac{d(\overrightarrow{O'M})}{dt}$$

It can be demonstrated that: $\frac{d(\overrightarrow{O'M})}{dt}\Big|_R = \frac{d(\overrightarrow{O'M})}{dt}\Big|_{R'} + \vec{\omega}\wedge\overrightarrow{O'M}\Big|_{R'}$

Given a vector $\vec{A}$ in the two Cartesian Coordinates system $(R_a)$and (R_r):

$$\vec{A}\big|_R = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

$$\vec{A}\big|_{R'} = x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}$$

$$\frac{d\vec{A}}{dt}\Big|_R = \frac{d}{dt}(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'})\Big|_R = \dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'} + \left[x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}\right]$$

We have seen that: $\frac{d\vec{\imath'}}{dt} = \vec{\omega}\wedge\vec{\imath'}$, $\frac{d\vec{\jmath'}}{dt} = \vec{\omega}\wedge\vec{\jmath'}$, $\frac{d\vec{k'}}{dt} = \vec{\omega}\wedge\vec{k'}$

$$\frac{d\vec{A}}{dt}\Big|_R = \dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'} + \left[x'\vec{\omega}\wedge\vec{\imath'} + y'\vec{\omega}\wedge\vec{\jmath'} + z'\vec{\omega}\wedge\vec{k'}\right]$$

$$\Rightarrow \boxed{\frac{d\vec{A}}{dt}\Big|_R = \frac{d\vec{A}_{/R}}{dt}\Big|_{R'} + \vec{\omega}\wedge\vec{A}_{/R'}}$$

Let's return to the acceleration relation:

$$\vec{a}_a = \ddot{x}'\vec{\imath'} + \ddot{y}'\vec{\jmath'} + \ddot{z}'\vec{k'} + \ddot{x}_0\vec{\imath'} + \ddot{y}_0\vec{\jmath'} + \ddot{z}_0\vec{k'} + \dot{x}'\frac{d\vec{\imath'}}{dt} + \dot{y}'\frac{d\vec{\jmath'}}{dt} + \dot{z}'\frac{d\vec{k'}}{dt} + \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M} + \vec{\omega}\wedge\frac{d(\overrightarrow{O'M})}{dt}$$

$$\vec{a}_a = \vec{a}_r + \frac{d^2\overrightarrow{OO'}}{dt^2} + \dot{x}'\frac{d\vec{\imath'}}{dt} + \dot{y}'\frac{d\vec{\jmath'}}{dt} + \dot{z}'\frac{d\vec{k'}}{dt} + \dot{\vec{\omega}}\wedge\overrightarrow{O'M} + \vec{\omega}\wedge\left(\frac{d(\overrightarrow{O'M})}{dt}\Big|_{R'} + \vec{\omega}\wedge\overrightarrow{O'M}\Big|_{R'}\right)$$

We have: $\dot{x}'\frac{d\vec{\imath'}}{dt}+\dot{y}'\frac{d\vec{\jmath'}}{dt}+\dot{z}'\frac{d\vec{k'}}{dt}=\dot{x}'\vec{\omega}\wedge\vec{\imath'}+\dot{y}'\vec{\omega}\wedge\vec{\jmath'}+\dot{z}'\vec{\omega}\wedge\vec{k'}=\vec{\omega}\wedge\left(\dot{x}'\vec{\imath'}+\dot{y}'\vec{\jmath'}+\dot{z}'\vec{k'}\right)=\vec{\omega}\wedge\frac{d\overrightarrow{O'M}}{dt}\Big|_{R'}$

$$\Rightarrow \vec{a}_a=\vec{a}_r+\frac{d^2\overrightarrow{OO'}}{dt^2}+\vec{\omega}\wedge\frac{d\overrightarrow{O'M}}{dt}+\dot{\vec{\omega}}\wedge\overrightarrow{O'M}+\vec{\omega}\wedge\left(\frac{d\overrightarrow{O'M}}{dt}\Big|_{R'}+\vec{\omega}\wedge\overrightarrow{O'M}\Big|_{R'}\right)$$

$$=\vec{a}_r+\frac{d^2\overrightarrow{OO'}}{dt^2}+\vec{\omega}\wedge\frac{d\overrightarrow{O'M}}{dt}\Big|_{R'}+\dot{\vec{\omega}}\wedge\overrightarrow{O'M}+\vec{\omega}\wedge\frac{d\overrightarrow{O'M}}{dt}\Big|_{R'}+\vec{\omega}\wedge(\vec{\omega}\wedge\overrightarrow{O'M}\Big|_{R'})$$

$$=\vec{a}_r+\frac{d^2\overrightarrow{OO'}}{dt^2}+\dot{\vec{\omega}}\wedge\overrightarrow{O'M}+\vec{\omega}\wedge(\vec{\omega}\wedge\overrightarrow{O'M}\Big|_{R'})+2.\vec{\omega}\wedge\frac{d\overrightarrow{O'M}}{dt}\Big|_{R'}$$

So: $\boxed{\vec{a}_a=\vec{a}_r+\vec{a}_e+\vec{a}_c}$

With: $\vec{a}_r=\vec{a}_{M/R}=\ddot{x}'\vec{\imath'}+\ddot{y}'\vec{\jmath'}+\ddot{z}'\vec{k'}$

$\vec{a}_r$ is the relative acceleration;

$$\vec{a}_e=\frac{d^2\overrightarrow{OO'}}{dt^2}+\dot{\vec{\omega}}\wedge\overrightarrow{O'M}+\vec{\omega}\wedge(\vec{\omega}\wedge\overrightarrow{O'M}\Big|_{R'})$$

$\vec{a}_e$ is the training acceleration (speed of R_r relative to R_a);

$$\vec{a}_c=2.\vec{\omega}\wedge\frac{d\overrightarrow{O'M}}{dt}\Big|_{R'}$$

$\vec{a}_c$ is the Coriolis acceleration or complementary acceleration.

Application

Let $R(O, x, y, z)$ be an absolute referential assumed to be Galilean provided with the direct orthonormal base $(\vec{\imath}, \vec{\jmath}, \vec{k})$ and $R'(O', x', y', z')$ a relative referential provided with the direct orthonormal base $(\vec{\imath'}, \vec{\jmath'}, \vec{k'})$. Over time, the (Oz) and (Oz') axes remain collinear.

In the vertical plane xOy, a circular rod with center C and radius r is kept fixed. A ring considered as a material point M slides without friction on the circular rod and it is identified in the relative reference by:

$\overrightarrow{O'M} = 2r \sin\theta \, \vec{\imath'}$

Where; $\theta = (\widehat{\vec{\imath}, \overrightarrow{O'M}})$.

1/ Check that the rotational speed of R in relation to R is given by $\vec{\omega}_{R'/R} = \dot{\theta}\vec{k}$.

2-a Calculate $\vec{v}_r(M)$ and $\vec{v}_a(M)$ respectively the relative and absolute velocities of M.

b- Deduce from it $\vec{U}_T$ the vector tangent to the trajectory.

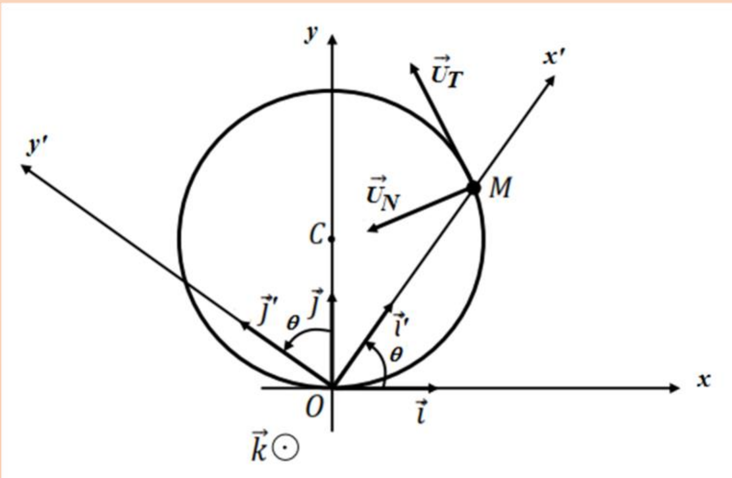
c- Determine $\vec{U}_N$ the normal vector to the trajectory.

3/ Determine $\vec{a}_r$ the relative acceleration of M.

4/ Determine $\vec{a}_e$ the training acceleration of R'.

5/ Determine $\vec{a}_c$ the Coriolis acceleration of M.

6/ Deduce $\vec{a}_a$ the absolute acceleration of M.



2-a/ Calculate of $\vec{v}_r(M)$:

$$\vec{v}_r(M) = \frac{d\overrightarrow{O'M}}{dt}\Big|_{R'} = \frac{d(2\,r\sin\theta\,\vec{\imath}'\,)}{dt} = 2r\dot{\theta}\cos\theta\,\vec{\imath}'$$

$\vec{\imath}'$ is constant in respect to R'.

$$\boxed{\vec{v}_r(M) = 2r\dot{\theta}\cos\theta\,\vec{\imath}'}$$

Calculate of $\vec{v}_a(M)$:

We have the law of speed composition: $\vec{v}_a = \vec{v}_r + \vec{v}_e$

$\vec{v}_e$ is the training speed; the speed of R' in relation to R:

$$\vec{v}_e = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}_{R'/R} \wedge \overrightarrow{O'M}$$

$\frac{d\overrightarrow{OO'}}{dt} = \vec{0}$ because $O \equiv O'$.

$$\vec{v}_e = \vec{\omega}_{R'/R} \wedge \overrightarrow{O'M} = \dot{\theta}\vec{k} \wedge 2\,r\sin\theta\,\vec{\imath}' = \dot{\theta}2\,r\sin\theta\;\vec{k}\wedge\vec{\imath}' = \dot{\theta}2\,r\sin\theta\,\vec{\jmath}'$$

$$\boxed{\vec{v}_e(M) = \dot{\theta}2\,r\sin\theta\,\vec{\jmath}'}$$

$$\vec{v}_a = 2r\dot{\theta}\cos\theta\,\vec{\imath}' + \dot{\theta}2\,r\sin\theta\,\vec{\jmath}' \Rightarrow \boxed{\vec{v}_a = 2r\dot{\theta}(\cos\theta\,\vec{\imath}' + \sin\theta\,\vec{\jmath}')}$$

b- Calculate of $\vec{U}_T$:

$\vec{U}_T$ is always tangential to the path so, it is parallel to $\vec{v}_a$.

$$\vec{U}_T = \frac{\vec{v}_a}{\|\vec{v}_a\|} = \frac{2r\dot{\theta}(\cos\theta\,\vec{\imath}' + \sin\theta\,\vec{\jmath}')}{2r\dot{\theta}.\sqrt{(\cos\theta)^2 + (\sin\theta)^2}} = \cos\theta\,\vec{\imath}' + \sin\theta\,\vec{\jmath}'$$

c- Calculate of $\vec{U}_N$:

$$\vec{U}_N = \frac{d\vec{U}_T}{d\theta} = -\sin\theta\,\vec{\imath}' + \cos\theta\,\vec{\jmath}'$$

3/ Determination of $\vec{a}_r$:

$$\vec{a}_r = \frac{d\vec{v}_r}{dt} = \frac{d(2r\dot{\theta}\cos\theta\,\vec{\imath}')}{dt} = 2r\ddot{\theta}\cos\theta\,\vec{\imath}' - 2r\dot{\theta}^2\sin\theta\,\vec{\imath}'$$

$$\boxed{\vec{a}_r = 2r(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\vec{\imath}'}$$

4/ Determination of $\vec{a}_e$:

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt^2} + \frac{d\vec{\omega}_{R'/R}}{dt} \wedge \overrightarrow{O'M} + \vec{\omega}_{R'/R} \wedge (\vec{\omega}_{R'/R} \wedge \overrightarrow{O'M})$$

$$\vec{a}_e = \vec{0} + \ddot{\theta}\vec{k} \wedge 2\,r\sin\theta\,\vec{\imath}' + \dot{\theta}\vec{k} \wedge \left(\dot{\theta}\vec{k} \wedge 2\,r\sin\theta\,\vec{\imath}'\right) = 2r\ddot{\theta}\sin\theta\,\vec{k}\wedge\vec{\imath}' +$$
$$\dot{\theta}\vec{k} \wedge \left(\dot{\theta}\;2\,r\sin\theta\,\vec{k}\wedge\vec{\imath}'\right) = 2r\ddot{\theta}\sin\theta\,\vec{\jmath}' + \dot{\theta}\vec{k} \wedge (\dot{\theta}\;2\,r\sin\theta\,\vec{\jmath}')$$

$\vec{a}_e = 2r\ddot{\theta}\sin\theta\,\vec{j'} - 2r\dot{\theta}^2\sin\theta\,\vec{\imath'}$

$\Rightarrow \boxed{\vec{a}_e = 2r\sin\theta\,(\ddot{\theta}\vec{j'} - \dot{\theta}^2\vec{\imath'})}$

5/ Determination of $\vec{a}_c$:

$\vec{a}_c = 2\,\vec{\omega}_{R'/R} \wedge \vec{v}_r = 2\dot{\theta}\vec{k} \wedge 2r\dot{\theta}\cos\theta\,\vec{\imath'}$

$\Rightarrow \boxed{\vec{a}_c = 4r\dot{\theta}^2\cos\theta\,\vec{j'}}$

4/ Determination of $\vec{a}_e$:

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt^2} + \frac{d\vec{\omega}_{R'/R}}{dt} \wedge \overrightarrow{O'M} + \vec{\omega}_{R'/R} \wedge (\vec{\omega}_{R'/R} \wedge \overrightarrow{O'M})$$

$\vec{a}_e = \vec{0} + \ddot{\theta}\vec{k} \wedge 2\,r\sin\theta\,\vec{\imath'} + \dot{\theta}\vec{k} \wedge \left(\dot{\theta}\vec{k} \wedge 2\,r\sin\theta\,\vec{\imath'}\right) = 2r\ddot{\theta}\sin\theta\,\vec{k} \wedge \vec{\imath'} +$
$\dot{\theta}\vec{k} \wedge \left(\dot{\theta}\,2\,r\sin\theta\,\vec{k} \wedge \vec{\imath'}\right) = 2r\ddot{\theta}\sin\theta\,\vec{j'} + \dot{\theta}\vec{k} \wedge (\dot{\theta}\,2\,r\sin\theta\,\vec{j'})$

$\vec{a}_e = 2r\ddot{\theta}\sin\theta\,\vec{j'} - 2r\dot{\theta}^2\sin\theta\,\vec{\imath'}$

$\Rightarrow \boxed{\vec{a}_e = 2r\sin\theta\,(\ddot{\theta}\vec{j'} - \dot{\theta}^2\vec{\imath'})}$

5/ Determination of $\vec{a}_c$:

$\vec{a}_c = 2\,\vec{\omega}_{R'/R} \wedge \vec{v}_r = 2\dot{\theta}\vec{k} \wedge 2r\dot{\theta}\cos\theta\,\vec{\imath'}$

$\Rightarrow \boxed{\vec{a}_c = 4r\dot{\theta}^2\cos\theta\,\vec{j'}}$

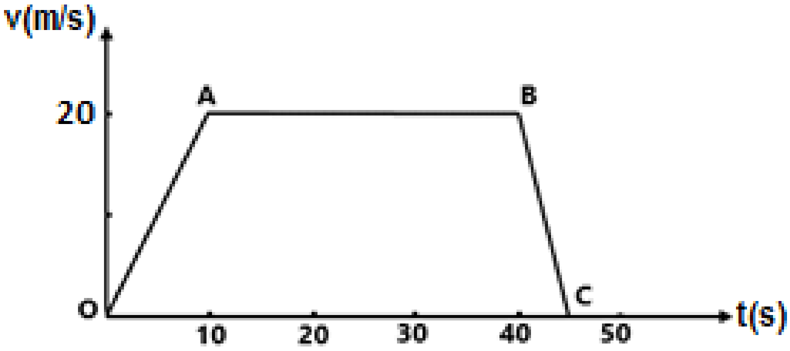
Set of exercises

(Kinematics of a material point)

Set of exercises (Kinematics of a material point)

Exercise 1:

The diagram below shows the velocity-time graph of a car that accelerates uniformly from rest with the acceleration 2 ms^{-2} for 10 s. It maintains a steady velocity for 30 s and is then brought to rest in 5 s under a uniform retardation.



1/ Find the equation of motion and the speed equation at each stage: OA, AB and BC, then find the distance traveled at every stage.
Conclude the total distance travelled.

Exercise 2:

A particle moves in a parabolic path defined by the position vector: $\vec{\boldsymbol{r}}(\boldsymbol{t}) = \boldsymbol{t}^2\vec{\boldsymbol{\imath}} + \sqrt{\boldsymbol{5} - \boldsymbol{t}^2}\vec{\boldsymbol{\jmath}}$
Where t is time in seconds.

1/ Find the velocity vector and its magnitude as functions of time.

2/ Show that the acceleration vector is written in the form: $\vec{\boldsymbol{a}} = \boldsymbol{2}\vec{\boldsymbol{\imath}} - \boldsymbol{5}(\boldsymbol{5} - \boldsymbol{t}^2)^{-\frac{3}{2}}\vec{\boldsymbol{\jmath}}$

Exercise 3:

A material point M moves in the plane $(\overrightarrow{Ox}, \overrightarrow{Oy})$ according to the time equations:

$$\begin{cases} \boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{2t} \\ \boldsymbol{y}(\boldsymbol{t}) = -\boldsymbol{5t}^2 + \boldsymbol{4t} \end{cases}$$

1/ Find the trajectory equation, what is it its nature? , plot it then locate the starting point of motion.

2/ Write the expression of $\overrightarrow{OM}$ vector position.

3/ Find and plot the velocity vector in two cases:

a - At the top of the trajectory.

b- When the trajectory intersects with the Ox axis.

4/ Write the expressions $\vec{\boldsymbol{v}}(\boldsymbol{t})$ and $\vec{\boldsymbol{a}}(\boldsymbol{t})$ in the curved coordinate system $(\overrightarrow{\boldsymbol{U_T}}, \overrightarrow{\boldsymbol{U_N}})$. What are the components of unit vectors $\overrightarrow{\boldsymbol{U_T}}$ and $\overrightarrow{\boldsymbol{U_N}}$ in the Cartesian coordinates system?

Exercise 4:

A particle moves in a path defined by the position vector: $\vec{\boldsymbol{r}}(\boldsymbol{t}) = \boldsymbol{t^2}\vec{\boldsymbol{\imath}} + (\boldsymbol{2t} - \boldsymbol{3})\vec{\boldsymbol{\jmath}} + (\boldsymbol{3t^2} - \boldsymbol{3t})\vec{\boldsymbol{\jmath}}$

Where t is time in seconds.

1/ Find tangential component $\vec{a}_T$ and normal component $\vec{a}_N$ of acceleration.

2/ Verify that we can find the same results using the expressions: $\boldsymbol{a_T} = \frac{\vec{v}.\vec{a}}{\|\vec{v}\|}$, $\boldsymbol{a_N} = \frac{\|\vec{v}\wedge\vec{a}\|}{\|\vec{v}\|}$

Exercise 5:

A material point **M** moves in the plane $(\overrightarrow{\boldsymbol{Ox}}, \overrightarrow{\boldsymbol{Oy}})$ of the Cartesian coordinate system according to:

$$\begin{cases} \boldsymbol{x(t) = a\cos\omega t} \\ \boldsymbol{y(t) = b\sin\omega t} \end{cases}$$

Where a, b, and ω are positive values with a > b.

1/ What is the equation of the trajectory of point M? Plot it graphically.

2/ Give the position vector $\overrightarrow{\boldsymbol{OM}}$, then calculate the velocity vector and its magnitude, and the acceleration vector and its magnitude.

3/ Find the tangential and normal components of the acceleration.

4/ Write the expressions $\vec{v}(t)$ and $\vec{a}(t)$ in the curved coordinate system ($\vec{U}_T, \vec{U}_N$).

5/ What are the components of unit vectors $\vec{U}_T$ and $\vec{U}_N$ in the Cartesian coordinate system?

6/ Find the expression for the radius of curvature of the trajectory.

Exercise 6:

The motion of a material point M in the set of polar coordinates $(\boldsymbol{O}, \vec{\boldsymbol{U}}_{\boldsymbol{\rho}}, \vec{\boldsymbol{U}}_{\boldsymbol{\theta}})$ is defined by the equations:

$$\begin{cases} \boldsymbol{\rho(t) = a\,t^2 + b} \\ \boldsymbol{\theta\,(t) = \omega t} \end{cases}$$

Where a, b, and ω are positive constants and t represents time.

1/ What are the units of the constants a, b and ω?

2/ What is the path equation?

3/ Calculate the $\vec{v}(t)$ and $\vec{a}(t)$ and their magnitudes, and deduce the unit vector, tangential to the path in the polar coordinates system.

4/ We consider the case:: $a = 1$, $b = 2$ and $\omega = \pi$

Draw the path of the point M in this case and then determine:

a- The location of point M at the following times: t = 1 s, t = 2 s and t = 2.5 s.

b- The initial velocity vector, plot it on the path.

c- The velocity vector when $t = 2\ s$ and the acceleration vector when $t = 2.5\ s$ and plot them on the path.

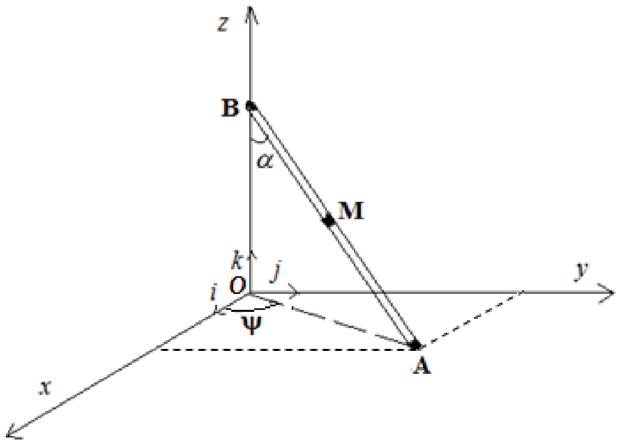
Exercise 7:

A material point M moves on a line segment AB of length l by the equation: $x' = \overline{AM} = \beta t$.

The point B is fixed on the Oz axis and A is in uniform rotation in the xOy plane around Oz (figure). We define the angle φ by: $\varphi = \left(\widehat{Ox, OA}\right) = \omega t$.

Calculate the absolute speed and acceleration of point M with respect to the $(Oxyz)$ reference.

β and ω are constants, α is the angle between $\overrightarrow{BO}$ and $\overrightarrow{BA}$.



Chapter III:

Dynamics of the material point

Chapter III: Dynamics of the material point

In the kinematics chapter we studied the characteristics of movement in the different coordinate systems, without referring to the causes of movement. In this chapter we are interested in the study of the movement of the material point which is the result of actions or forces of classical mechanics (Newton's laws).

Moving a material point or dynamic of a material point is the study of forces and its effect on the material point, with the possibility of with predictability of movement.

Isaac Newton has developed dynamic laws that apply in Galilean references.

III.1 Galilean (inertial) reference

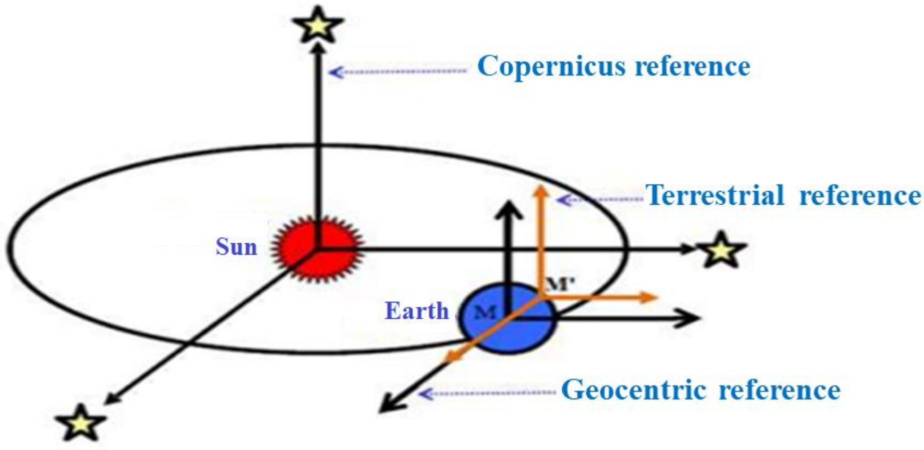
It is every reference that achieves the principle of inertia (Newton's first law), meaning that the reference must be in a immobile state or in a state of uniform rectilinear motion, In the contrary case, then the reference is not Galilean.

So the study of the movement of a system is done with respect to a static reference or in a state of uniform rectilinear motion.

The motion is studied in Galilean references, and this depends on the issue raised, examples of these references:

Copernicus reference:

Its origin is the center of the solar system, their three axes are directed towards three fixed stars with respect to the sun.



Kepler (or heliocentric) reference

Its origin is the center of inertia of the sun, their axes are oriented like those of the Copernican one.

Geocentric reference

Its origin is the center of the earth and its axes are parallel to those of the Copernican one. These three references are used to study the motion of planets.

Terrestrial reference

Its origin is a point on the surface of the earth, its axes are fixed with respect to the earth.

This reference is used for the study of objects in motion on the earth (or linked to it).

The earth is considered fixed, and it therefore represents a Galilean reference. For example, it uses to the study of an airplane movement, a car, etc.

III.2 Newton's laws of motion

III.2.1 Newton's first law: principle of inertia

In the Galilean reference, if a material point is isolated, that is, not subject to any external force, or semi-isolated, that is, the resultant of the forces subject to it is non-existent, then it is immobile or in a state of uniform rectilinear motion.

$$\boxed{\sum \vec{F}_{ext} = \vec{0}} \Rightarrow \vec{v} = \overrightarrow{cte}$$

The sum of the external forces acting on the material point is zero.

III.2.2 Newton's second law

Newton's second law represents the fundamental principle of dynamics (FPD), it relates the acceleration of a material point to the external applied forces.

Announcement: In a Galilean reference, the sum of the external applied forces to a material point is proportional to its acceleration vector $\vec{a}$, the proportionality coefficient is the mass m of the body in motion.

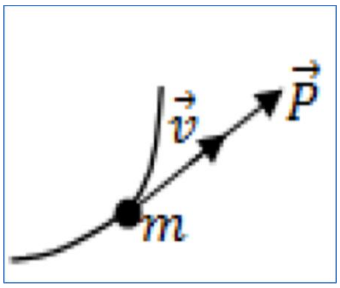
$$\boxed{\sum \vec{F}_{ext} = m\vec{a}}$$

The Momentum

Each material point with mass m and moving with speed $\vec{v}$ has a momentum $\vec{P}$, which is given by:

$$\vec{P} = m\vec{v}$$

$\vec{P}$ and $\vec{v}$ are parallel



Momentum theorem

In a Galilean reference, the derivative, with respect to time, of the momentum of a material point is equal to the sum of the external applied forces.

$$\left.\frac{d\vec{P}}{dt}\right|_{RG} = \sum \vec{F}_{ext}$$

$\frac{d\vec{P}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt}$; $m = Cte \Rightarrow m\vec{a} = \sum \vec{F}_{ext}$

The momentum theorem then represents Newton's second law. And we can give a new statement of the principle of inertia: "a free particle always moves with a constant momentum".

Notion of inertia

Inertia or mechanical inertia is the resistance to any variation in speed and/or in direction.

For a material point, inertia is represented by a positive scalar called mass and denoted by m. If the mass of the material point is greater, its more difficult to change its speed.

Notion of the force

The force is an action applied on a body can lead to its acceleration, i.e it modifies the intensity (modulus) and/or the direction of its velocity. We give the examples:

$\vec{g}$: the acceleration of gravity

$\vec{p}$: the weight of a body

$\vec{F}$: the electrostatic force (which is the force resulting between two charges) …

In the international system of units SI, the force is expressed in Newtons (N). The Newton is deduced from the formula,

$$\sum \vec{F}ext = m\vec{a}$$

and is given by: $1N = kg.m/s^2$

Example:

A body with a mass of 10 kg, acted on to the force F= 120t+40 N. At time t=0, x = 5m and v = 6 m/s.

Find the speed and the position of this mobile at time t.
Solution:

$\sum \vec{F}_{ext} = m\vec{a} \Rightarrow F = ma \Rightarrow 120t + 40 \Rightarrow a = \frac{120}{10}t + \frac{40}{10} = 12t + 4. \Rightarrow \boxed{a = 12t + 4}$

$a = \frac{dv}{dt} \Rightarrow dv = adt \Rightarrow \int dv = \int adt = \int (12t + 4)dt.$

$$\Rightarrow v = \frac{12}{2}t^2 + 4t + c \Rightarrow v = 6t^2 + 4t + c\,.$$

$at\ t = 0\,, v = 6 \Rightarrow 6 = 6 \times 0 + 4 \times 0 + c \Rightarrow c = 6\,.$

$\Rightarrow \boxed{v = 6t^2 + 4t + 6}.$

$v = \frac{dx}{dt} \Rightarrow dx = vdt \Rightarrow \int dx = \int vdt = \int (6t^2 + 4t + 6)dt.$

$\Rightarrow x = \frac{6}{3}t^3 + \frac{4}{2}t^2 + 6t + \grave{c} = 2t^3 + 2t^2 + 6t + C'.$

$at\ t = 0\,, x = 5 = 2 \times 0 + 2 \times 0 + 6 \times 0 + C' \Rightarrow C' = 5\,.$

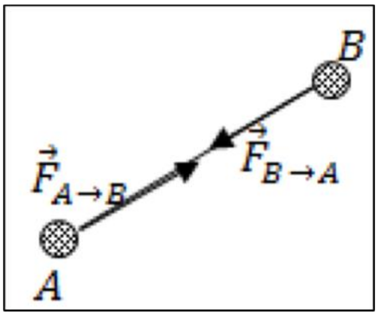
$$\Rightarrow \boxed{x(t) = 2t^3 + 2t^2 + 6t + 5}$$

III.2.3 Newton's third law

The law of action and reaction.

Statement: If a body A acts on a body B with a force $\vec{F}_{A\to B}$,then body B acts on the body A with a force $\vec{F}_{B\to A}$ which has the same modulus as $\vec{F}_{A\to B}$ but in the opposite direction.

$$\boxed{\vec{F}_{A\to B} = -\vec{F}_{B\to A}}$$



Principle of momentum conservation

The momentum of an isolated or pseudo-isolated system is a conservative quantity.

In a Galilean reference frame if a material system consists of two material points A and B and if the system is isolated or pseudo-isolated. During a collision between A and B, the momentum of the system before and after the collision are equals (for a perfectly elastic collision).

$$\sum \vec{P}_{init} = \sum \vec{P}_{fin}$$

Note:

There are two types of collision:

- Elastic collision: the mobiles A and B bounce off each other and there is conservation of kinetic energy.
- Inelastic collision: the mobiles remain stuck at the moment of impact and then move together in this case the total kinetic energy is not conserved.

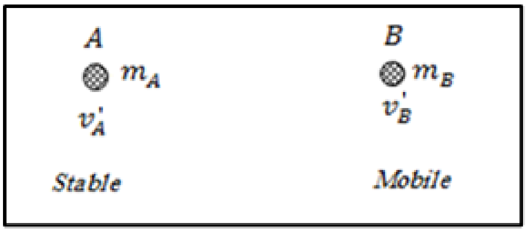
In the case of elastic collision momentum is conserved and we have

Before collision:

$$\vec{P}_{initial} = \vec{P}_A + \vec{P}_B = m_A\vec{v}_A + m_B\vec{v}_B$$

$$m\vec{v}_B = \vec{0} \Rightarrow \vec{P}_i = m_A\vec{v}_A$$

After collision:



$$\vec{P}_{final} = \vec{P}_f = m_A\overrightarrow{v'}_A + m_B\overrightarrow{v'}_B$$

$$\Rightarrow \vec{P}_f = m_A\overrightarrow{v'}_A + m_B\overrightarrow{v'}_B$$

$$\Rightarrow \vec{P}_f = \vec{P}_i = m\vec{v}_A = m\overrightarrow{v'}_A + m_B\overrightarrow{v'}_B$$

Example:

Consider two balls A and B. A with mass $10\ kg$ *and B with mass* $5\ kg$.

The ball A had a velocity of $3\ m/s$ *before the collision, and it becomes* $2\ m/s$ *after the collision when it makes an angle equal to 30° with the ox axis.*

Calculate the speed of B after the collision and what is the angle direction of velocity vector.

Solution:

Before collision, the initial momentum Pi is;

$$\vec{P}_i = \vec{P}_{Ai} + \vec{P}_{Bi} = m\vec{v}_A + m_B\vec{v}_B; \quad v_B = 0\,, v_A = 3m/s$$

$$\vec{P}_A = m_A\vec{v}_A = \vec{P}_i = 10.3\,\vec{\imath} = 30\,\vec{\imath}$$

After collision:

$$\vec{P}_f = \vec{P}_{Af} + \vec{P}_{Bf}$$

$$(A):\ \vec{P}_{Af} = m_A v_{Ax}\vec{\imath} + m_A v_{Ay}\vec{\jmath}$$

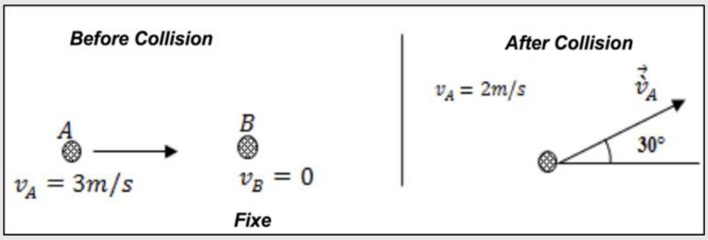
$$v_{Ax} = 2\,.\cos 30 = 2.\frac{\sqrt{3}}{2} = \sqrt{3}$$

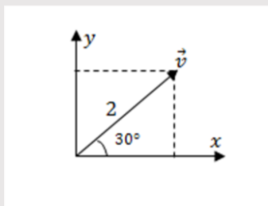
$$v_{Ay} = 2\,.\sin 30 = \frac{2}{2} = 1$$

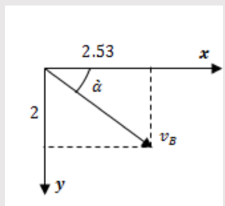
$$\Rightarrow \vec{P}_{Af} = m_A\sqrt{3}\vec{\imath} + m_A\vec{\jmath} \ \Rightarrow \vec{P}_{Af} = 10\sqrt{3}\vec{\imath} + 10\,\vec{\jmath}$$

$$(B):\ \vec{P}_{Bf} = m_B v_{Bx}\vec{\imath} + m_B v_{By}\vec{\jmath}$$

$$\Rightarrow \vec{P}_f = \vec{P}_{Af} + \vec{P}_{Af} = \left(10\sqrt{3} + m_B v_{Bx}\right)\vec{\imath} + (10 + m_B v_{By})\vec{\jmath}$$



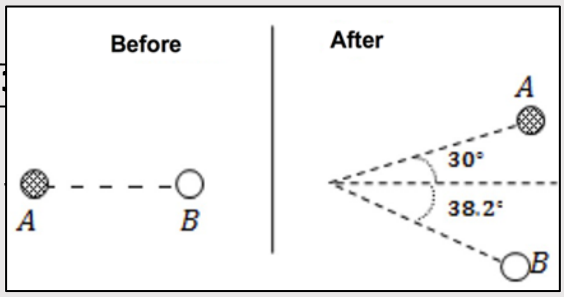




We apply momentum conservation law: $\vec{P}_f = \vec{P}_i$

$$30\,\vec{\imath} \ = \ \left(10\sqrt{3} + 5v_{Bx}\right)\vec{\imath} + \left(10 + 5v_{By}\right)\vec{\jmath}$$

$$\begin{cases} 30 = 10\sqrt{3} + 5v_{Bx} \\ 0 = 10 + 5v_{By} \end{cases} \Rightarrow \begin{cases} v_{Bx} = \dfrac{30 - 10\sqrt{}}{5} \\ v_{By} = -\dfrac{10}{5} = \end{cases}$$



$$\begin{cases} v_{Bx} = 2.53 \\ v_{By} = -2 \end{cases} \ \Rightarrow \ \vec{v}_B = 2,53\,\vec{\imath} - 2\,\vec{\jmath}$$

III.3 Interaction forces

The purpose of dynamics is to predict and to study the motions of bodies which are subjected to forces, some of these forces acting at a distance, like the gravitational force, and others acting in contact (close-acting) like the reaction force.

III.3.1 forces acting at a distance

Gravitational force

Two massive bodies attract each other with two forces having the same absolute value (intensity) but with opposite directions. The modulus of this force is proportional to the product of the two masses, and it is inversely proportional to the square of the distance separating them and connecting their centers of mass.

$$\vec{\boldsymbol{F}}_{\boldsymbol{M}\to\boldsymbol{m}} = -\boldsymbol{G}\frac{\boldsymbol{Mm}}{\boldsymbol{r}^2}\vec{\boldsymbol{u}}$$

***r**:* is the distance between the centers of masses $\boldsymbol{M}$ and $\boldsymbol{m}$.

***G**:* is the gravitational constant;

G = 6.67384 .10^{-11} $N\, m^2\, kg^{-2}$.

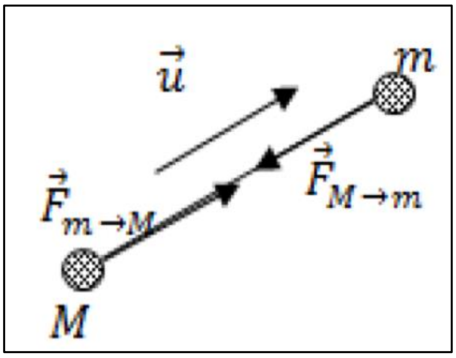
$\vec{\boldsymbol{u}}$: is a unit vector, it is directed from $\boldsymbol{M}$ to $\boldsymbol{m}$.

If $\boldsymbol{M}$ represents the mass of the earth and $\boldsymbol{m}$ located at a distance r from the center of the earth

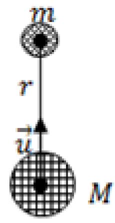
and

The vector $\vec{g}$ represents the gravitational field of the earth, it is always oriented towards its center.

$\vec{\boldsymbol{g}} = -\boldsymbol{G}\frac{\boldsymbol{M}}{\boldsymbol{r}^2}\vec{\boldsymbol{u}}\ ;\ \ \boldsymbol{g} = \boldsymbol{G}\frac{\boldsymbol{M}}{\boldsymbol{r}^2}$



The force $\vec{F}_{M\to m} = m\vec{g}$ represents the weight of the body ***m***, such that $\vec{P} = m\vec{g}$



In the vicinity of the earth (*r* = *R*):

R: is the radius of the earth,

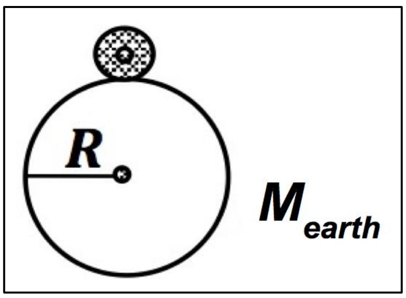
R = 6.371 10^6 km.

M

$$\|\vec{g}\| = G.\frac{M}{R^2} = \frac{6.67384.10^{-11}.5.9742.10^{24}}{(6.371.10^6)^2}$$

$$\|\vec{g}\| = 0.9822.10^{-11}.10^{24}.10^{-12} = g_0 = 9.80\ \frac{m}{s^2}$$

For a body on the surface of the earth, the constant g becomes g_0, and g_0 = 9.80 m.s^{-1}.



At a level r from the earth, we can write g :

$$g(h) = \frac{G.M}{(R+h)^2} = \frac{G.M}{(R\left(1+\frac{h}{R}\right))^2} = \frac{G.M}{R^2.\left(1+\frac{h}{R}\right)^2}$$

We have: $g_0 = \frac{G.M}{R^2} \Rightarrow g(h) = g_0 . \frac{1}{\left(1+\frac{h}{R}\right)^2} \Rightarrow g(h) = g_0 . \left(1+\frac{h}{R}\right)^{-2}$

If $h \ll R \Rightarrow \frac{h}{R} \ll 1$

The limited development of order 1 of: $(1+x)^n = 1 + nx$

$\Rightarrow \left(1+\frac{h}{R}\right)^{-2} = 1 - 2.\frac{h}{R} \Rightarrow g(h) = g_0 . \left(1 - 2.\frac{h}{R}\right).$

$\Rightarrow g(h) = g_0(1 - 3.1\times10^{-7}.h)$

$\vec{g} = -g_0(1 - 3.1\times10^{-7}.h)\,\vec{u}$

So, anybody located at a height h from the earth falls toward earth with the acceleration g(h) which does not depend on the mass of the body (free fall).

III.3.2 Contact force

Contact action: when two bodies are in contact, there appears either a reaction or a frictional force.

Normal reaction of a solid support

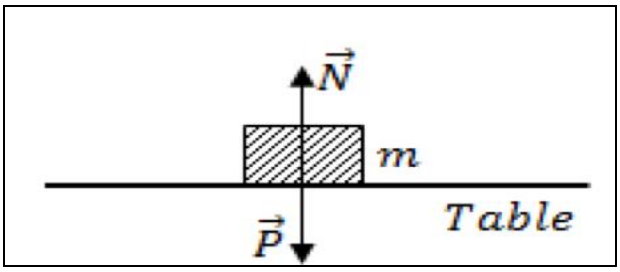
When a mass *m* is placed on a table (or a solid support) the weight of this mass acts on the table which exerts

a force equal and opposite to the weight.

This is the normal reaction of the table on *m*, it is denoted $\vec{N}$ (or sometimes $\vec{R}$):

$$\vec{N} = -\vec{P}$$

$$\vec{P} = m\,\vec{g}$$



Friction force

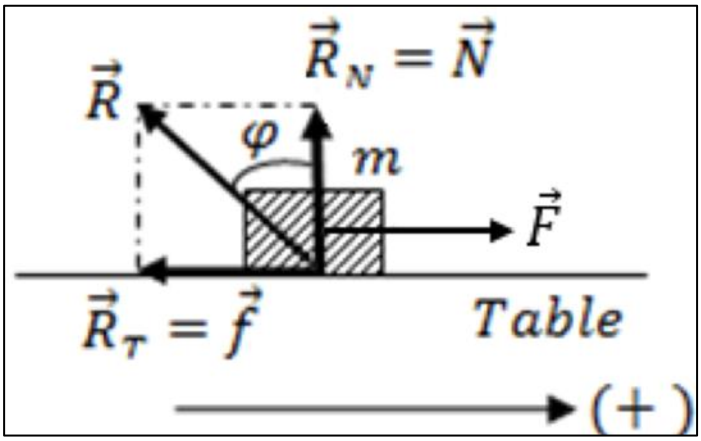
Frictional forces appear between a moving body and a support, this is the solid friction; or between a moving body in a fluid (liquid or gas), and this is the viscous friction.

If a body *A* is moving on a support *B*, the reaction force of *B* on *A* has two components:

$\vec{R}_N = \vec{N}$ the normal reaction to the support

$\vec{R}_T = \vec{f}$ the tangential reaction to the support

$\vec{R}_T$: Represent the solid frictional force $\vec{f}$, it is parallel to the direction of motion but in the opposite way. The sum of $\vec{R}_N$ and $\vec{R}_T$ is $\vec{R}$, the total reaction of the support on the body.



The body is fixed if, $\vec{f} = \overrightarrow{-F}$

$\|\vec{f}\|$: the modulus of the friction force is proportional to $\|\vec{N}\|$, the proportionality coefficient is μ and is named solide friction coefficient:

$$\mu = \frac{\|\vec{f}\|}{\|\vec{N}\|} = \frac{\|\vec{R}_T\|}{\|\vec{R}_N\|} = tg\ \varphi$$

φ: is the angle between$\vec{N}$ and $\vec{R}$.

μ is dimensionless (unitless), and it should be less than 1: $\mu < 1$

There are two types of solid friction: static and kinetic it is related to the nature of the surface:

Static solid friction

Suppose, a force $\vec{F}$ applied to push a body A.

For the body *A* not to move, a limiting frictional force $\vec{f}_{max}$ preventing its movement, is called the static friction force $\overrightarrow{f_s}$, for which corresponds the coefficient of static friction μ_s.

$$\mu_s \geq \frac{\|\vec{f}_{max}\|}{\|\vec{N}\|}$$

$$\Rightarrow \ \|\overrightarrow{f_s}\| = \|\vec{f}_{max}\| < \mu_s \,.\, \|\vec{N}\|$$

The body is fixed as long as the force of static friction is greater than the applied force.

Kinetic solid friction

When the body begins to move, the frictional force decreases to a certain value, the value of friction force of which the body can moves is called the kinetic frictional force $\overrightarrow{f_c}$.

$f_c = \mu_c \, N$

$\mu_c = \ tg \ \varphi_c = \frac{f_c}{N}$

Where: $\mu_c < \mu_s$

<u>Note</u>:

- The reaction N is always greater than zero, and if the body leaves the support the reaction will be zero N = 0.

- In general, μ_c is always less than 1.

Example:

A body of mass m=0.5 kg placed on a rough plane with an inclination angle θ.

The coefficient of static friction μs = 0.8.

1/ What must be the angle θ for the body to take off?

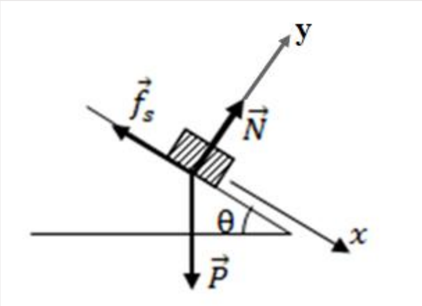
2/ What is the maximum static frictional force?

3/ for θ=35°;

- Calculate the normal reaction force.

- Calculate the force of static friction.

Solution:



1/ The body is immobile, we apply the first Newton's law:

$\sum \vec{F}_{ext} = \vec{0} \Rightarrow \vec{P} + \vec{N} + \vec{f_s} = \vec{0}$

By projecting forces onto the axes of motion Ox *and* Oy*:*

$$\begin{cases} Ox: P.\sin\theta - f_s = 0 \\ Oy: N - P.\cos\theta = 0 \end{cases} \Rightarrow \begin{cases} f_s = P.\sin\theta \\ N = P.\cos\theta \end{cases}$$

$$\mu_s \geq \frac{f_{max}}{N} = \frac{f_s}{N} \Rightarrow \mu_s \geq \frac{P.\sin\theta}{P.\cos\theta} = \tan\theta$$

$$\theta < \arctan(0{,}8) = 38{,}65°$$

2/ $f_{max} = f_s = P.\sin 38{,}65 = 0{,}5.10.\sin 38{,}65 = 3{,}12\ N. \Rightarrow f_s = 3{,}12\ N$

For $\theta = 35°$:

3/ $N = P.\cos\theta = 0{,}5.10.\cos 35° = 4{,}09\ N.$

4/ $f_s = P.\sin\theta = 5.\ \sin 35° = 2{,}86\ N.$

Viscous friction

When a body moves in a fluid (gas or liquid) with a low speed, they appear a frictional force $\vec{F}_f$ acting on the body and the fluid:

$$\vec{F}_f = \; -K\,\vec{v}$$

$\vec{v}$: is the velocity of the body.

K: is the coefficient of viscous friction.

In the case of a spherical body, the force of viscous friction becomes:

$$\vec{F}_f = \; -\; 6\pi R\mu\,\overrightarrow{v}$$

It's called the Stokes

With: R is the radius of the body, and μ is the dynamic viscosity of the fluid.

Note about viscous friction

If the speed of the moving body in a fluid is low, the friction force is called Stokes viscous friction force:

$$\vec{F}_f = -\,k\vec{v} \;\Rightarrow F_f = kv$$

$[k] = M.T^{-1}$.

For relatively high speeds, we then speak of Newton's friction force :

$$F_f = -\,kv^2 \;\;;\;\; [k] = M.L^{-1}$$

k: Is the coefficient of friction.

III.3.3 Elastic force

Elastic force is the force exerted by an object when it is deformed. This force can be caused by any object that can be deformed, not just springs. For example, an elastic force can be caused by a flexible object such as rubber. In this lesson we are interested by the elastic force of a spring.

The Spring:

The spring is characterized by its initial length $l0$ (equilibrium length) and its stiffness constant k (coefficient of expansion) , which depends on the material of the spring and its shape.

$[\mathrm{k}] = \mathrm{M.T^{-2}}$

A spring has always the tendency to return to its initial state after its elongation or its compression**.**

Hooke's law

If x is the extension of the spring, then the spring applies a force to return to its equilibrium position, this force is called the restoring force it is usually symbolized by: $\vec{F}_r$ and it is proportional to the elongation x.

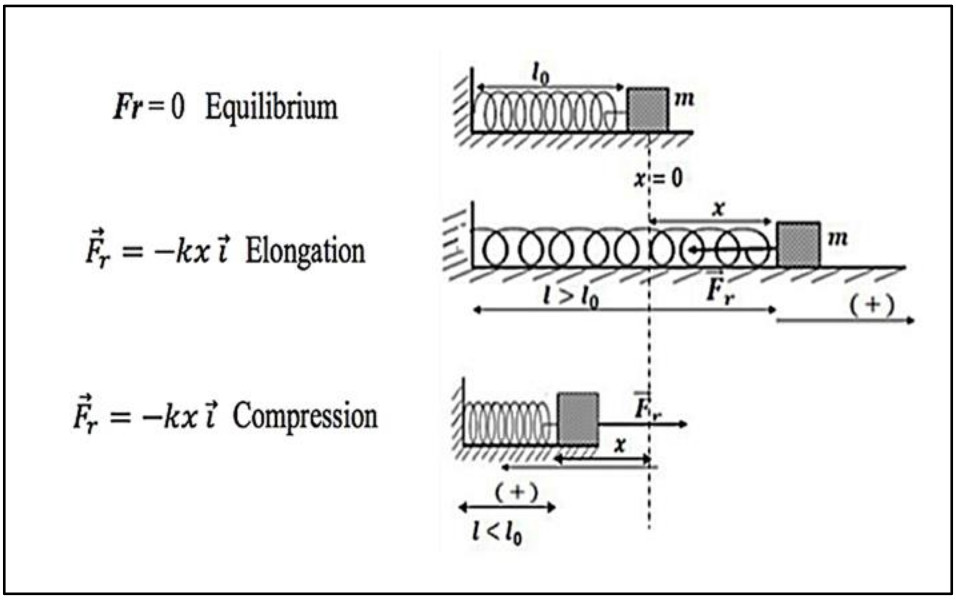
Hooke's law gives the relation of $\vec{F}_r$: $F_r = -kx$

$\vec{F}_r$ is always oriented in the opposite direction of the motion.

x: is the extension of the spring: $x = l - l_0$.

l_0: Initial length of the spring.

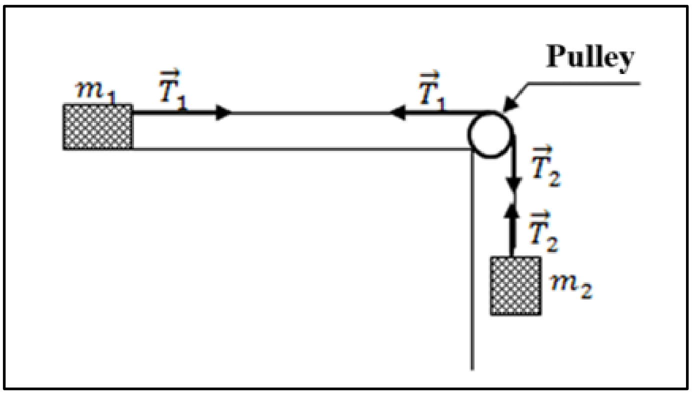
l: Length of moving spring.



III.3.4 Tension force

Tension force is a pulling force that acts along the length of stretched flexible connector, such as a rope or cable. It is parallel to its length.

If the rope is inextensible and has a negligible mass, the tension force is constant, and each point of the rope is subject to two equal and opposite forces (see figure below).



III.4 Dynamics of rotational motion

III.4.1 Momentum of a force

The momentum of force $\overrightarrow{\mathcal{M}}(\vec{F})$ is a vectorial quantity that represents the ability of a force to rotate an object around a point or axis. It is defined as the vector product between the arm (the distance between the force support and the point of rotation) and the force. A force $\vec{F}$ is applied to a material point M, then the momentum of the force $\vec{F}$ to rotate M around point A constant is:

$$\boxed{\overrightarrow{\boldsymbol{\mathcal{M}}}_{/A}(\vec{\boldsymbol{F}}) = \overrightarrow{\boldsymbol{AM}} \wedge \vec{\boldsymbol{F}}}$$

$$\|\overrightarrow{\boldsymbol{\mathcal{M}}}_{/A}(\vec{\boldsymbol{F}})\| = \|\overrightarrow{\boldsymbol{AM}}\|.\|\vec{\boldsymbol{F}}\|.\boldsymbol{sin\,\alpha}$$

α is the angle between $\overrightarrow{AM}$ and $\vec{F}$.

III.4.2 Angular momentum

We define the angular momentum $\mathcal{L}_{/A}$ of a material point M with respect to a fixed point (or an axis) A by the moment of its momentum P:

$$\boxed{\vec{\boldsymbol{\mathcal{L}}}_{/A} = \overrightarrow{\boldsymbol{AM}} \wedge \vec{\boldsymbol{P}}}$$

III.4.3 Theorem of angular momentum

In a Galilean reference, the derivative with respect to the time of the angular momentum of a material point with respect to a fixed point, is equal to the moment of the resultant of the external applied forces to the point M with respect to a fixed axis (passing through A).

$$\boxed{\frac{\boldsymbol{d}\,\vec{\boldsymbol{\mathcal{L}}}_{/A}}{\boldsymbol{dt}} = \sum \overrightarrow{\boldsymbol{\mathcal{M}}}_{/A}(\vec{\boldsymbol{F}}_{\boldsymbol{ext}})}$$

We have: $\vec{\mathcal{L}}_{/A} = \overrightarrow{AM} \wedge \vec{P}$

$\vec{P} = m\vec{v}$: Momentum

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\vec{\boldsymbol{\mathcal{L}}}_{/A} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}(\overrightarrow{\boldsymbol{AM}} \wedge \vec{\boldsymbol{P}}) = \frac{\boldsymbol{d}\overrightarrow{\boldsymbol{AM}}}{\boldsymbol{dt}} \wedge \vec{\boldsymbol{P}} + \overrightarrow{\boldsymbol{AM}} \wedge \frac{\boldsymbol{d}\vec{\boldsymbol{P}}}{\boldsymbol{dt}}$$

In the Galilean frame of reference R we have:

$$\overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OM} = \overrightarrow{OM} - \overrightarrow{OA}$$

$$\Rightarrow \frac{d}{dt}\vec{\mathcal{L}}_{/A} = \frac{d}{dt}\left(\overrightarrow{OM} - \overrightarrow{OA}\right) \wedge \vec{P} + \overrightarrow{AM} \wedge \frac{d\vec{P}}{dt}$$

$$= \frac{d}{dt}\left(\overrightarrow{OM}\right) \wedge \vec{P} - \frac{d\overrightarrow{OA}}{dt} \wedge \vec{P} + \overrightarrow{AM} \wedge \frac{d\vec{P}}{dt}$$

And we have: $\frac{d}{dt}\left(\overrightarrow{OM}\right) \wedge \vec{P} = \vec{v} \wedge \vec{P} = \vec{v} \wedge m\vec{v} = \vec{0}$; Because $\vec{v} \,//\, \vec{P}$

$\frac{d\overrightarrow{OA}}{dt} = \vec{0}$, because A is immobil in R.

$\Rightarrow \; \frac{d}{dt}\vec{\mathcal{L}}_{/A} = \overrightarrow{AM} \wedge \frac{d\vec{P}}{dt}$

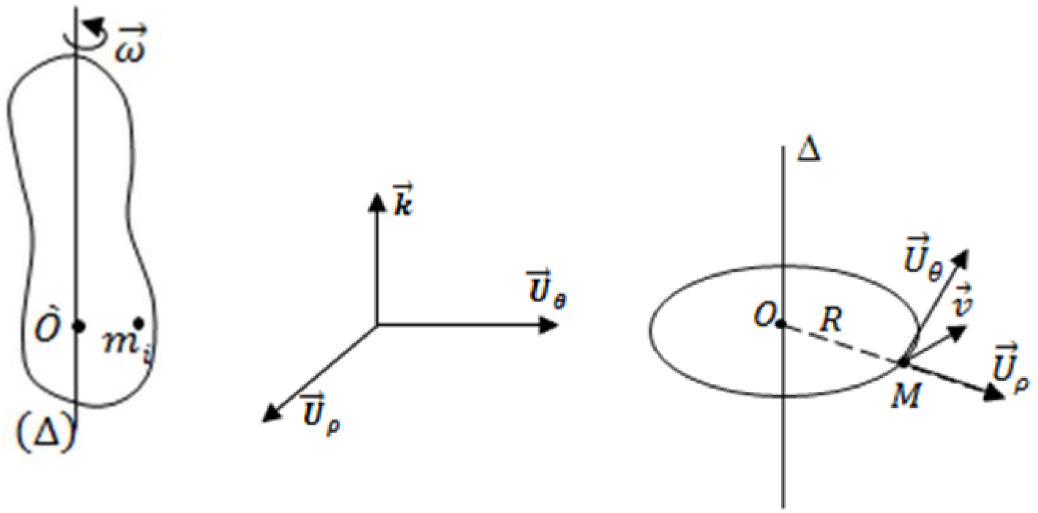
According to Newton's first law**:** $\frac{\boldsymbol{d}\vec{\boldsymbol{P}}}{\boldsymbol{dt}} = \sum \vec{\boldsymbol{F}}_{\boldsymbol{ext}}$

$\Rightarrow \boxed{\frac{\boldsymbol{d}}{\boldsymbol{dt}}\vec{\boldsymbol{\mathcal{L}}}_{/A} = \overrightarrow{\boldsymbol{AM}} \wedge \sum \vec{\boldsymbol{F}}_{\boldsymbol{ext}}}$

III.4.4 Application of the angular momentum theorem

We apply the angular momentum theorem to the rotation around a fixed axis. The solid body is made up of a number of material points; the i^th point is of mass m_i and occupying the point M_i of the body, the angular momentum of this point with respect to a point O on the axis (Δ) is:

$$\vec{\mathcal{L}}_i = \overrightarrow{OM} \wedge \vec{P}_i = \overrightarrow{OM} \wedge m_i \vec{v}_i$$



The kinematic parameters in the polar coordinates of the i^{th} point M_i are:

$$\overrightarrow{OM_i} = R.\vec{U}_\rho$$

$$\vec{v}_i = \frac{d\overrightarrow{OM_i}}{dt} = R_i \frac{d\vec{U}_\rho}{dt} = R_i\, \dot{\theta}.\vec{U}_\theta ; \vec{\omega} = \dot{\theta}\vec{k}$$

$$\vec{\mathcal{L}}_i = \overrightarrow{OM_i} \wedge \vec{P}_i = \overrightarrow{OM_i} \wedge m_i \vec{v}_i = R_i.\vec{U}_\rho \wedge m_i R_i \dot{\theta}.\vec{U}_\theta$$

$$= m_i R_i^2 \dot{\theta}\, \overrightarrow{U_\rho} \wedge \overrightarrow{U_\theta} = m_i R_i^2 \dot{\theta} \vec{k} = m_i R_i^2 \vec{\omega}$$

$$\vec{\mathcal{L}}_i = m_i R_i^2 \vec{\omega}$$

We assume that:

$$j_i = m_i R_i^2 \;\Rightarrow j_{/\Delta} = \textstyle\sum_i j_i = \sum_i m_i R_i^2$$

The sum is carried out over all the points of the solid body and becomes an integral over the volume or the surface of the solid:

$$\sum_i m_i R_i^2 = \iiint_V \rho R^2 dV$$

The density volume is $\rho = m/V$

$$\vec{\mathcal{L}}_i = j_i \vec{\omega}$$

$\Rightarrow \vec{\mathcal{L}}_{/A} = \sum j_i \vec{\omega}$

$\Rightarrow \boxed{\vec{\boldsymbol{\mathcal{L}}}_{/A} = \boldsymbol{j}_{/\Delta}\vec{\boldsymbol{\omega}}}$

$j_{/\Delta}$ Is a fixed quantity which characterizes the rotating body, it is its moment of inertia relative to the axis of rotation.

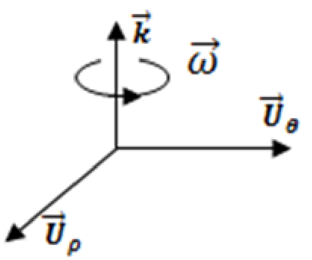
$$\frac{\mathrm{d}\vec{\mathcal{L}}_{/A}}{\mathrm{dt}} = \mathrm{j}_{/\Delta}\frac{\mathrm{d}\vec{\omega}}{\mathrm{dt}} = \mathrm{j}_{/\Delta}\vec{\ddot{\theta}}$$

And we have: $\frac{\mathrm{d}\vec{\mathcal{L}}_{/A}}{\mathrm{dt}} = \overrightarrow{\mathrm{AM}} \wedge \sum \vec{\mathrm{F}}_{\mathrm{ext}} = \sum \vec{\mathcal{M}}_{/\Delta}(\vec{\mathrm{F}}_{\mathrm{ext}})$

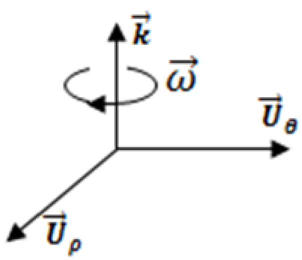
$\Rightarrow \boxed{\sum \vec{\boldsymbol{\mathcal{M}}}_{/\Delta}(\vec{\mathbf{F}}_{\mathbf{ext}}) = \mathbf{j}_{/\Delta}\vec{\ddot{\boldsymbol{\theta}}}}$

It can be said that that relation $\left(\sum \vec{\mathcal{M}}_{/\Delta}(\vec{F}_{ext}) = j_{/\Delta}\vec{\ddot{\theta}}\right)$ represents the basic principle of rotational motion.

Regarding the direction of the angular acceleration vector $\vec{\ddot{\theta}}$, it obeys the rule of the three fingers of the right hand:



$\vec{\ddot{\boldsymbol{\theta}}} = \ddot{\boldsymbol{\theta}}\vec{\boldsymbol{k}}$ if



and $\vec{\ddot{\boldsymbol{\theta}}} = -\ddot{\boldsymbol{\theta}}\vec{\boldsymbol{k}}$ if

Application:

A material point M with mass m moves along the path ABCD (Figure) from point A without initial speed.

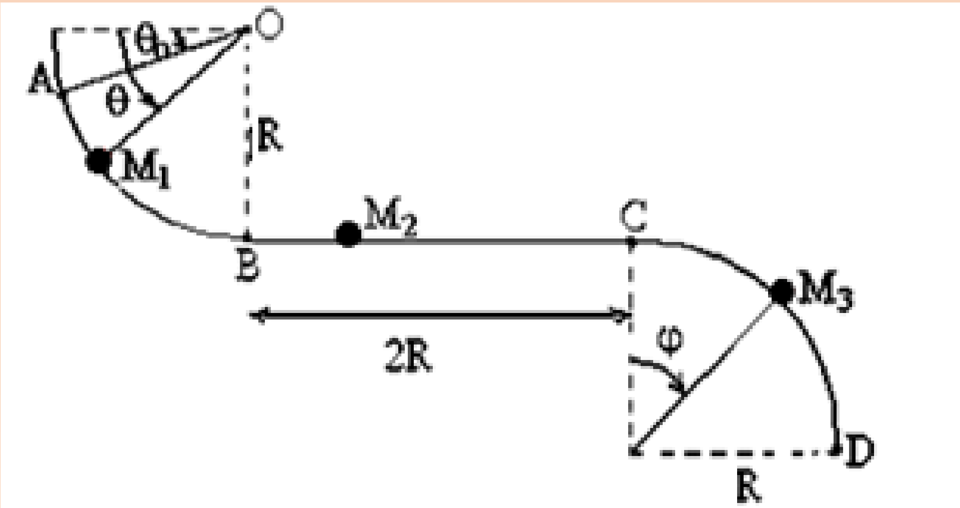
1/ In part AB we assume that there is no friction. Find the speed of the material point at point M_1.

2/ Find the speed with which the point M reaches the point C if the segment BC is smooth.

3/ Find this speed if there is a friction force F ⃗_f on the part BC with a constant coefficient of friction μ.

4/ We consider the friction to be negligible in part CD. Find the speed at point M_3.

5/ What is the angle at which the point M leaves the path?



Solution:

1/ Calculate of speed in point M_1.

We apply the fundamental principle of dynamics (the second Newton's law): $\sum \vec{F}_{ext} = m\vec{a} \Rightarrow \vec{P} + \vec{N} = m\vec{a}$

We use the intrinsic coordinates $(\overrightarrow{U_T}, \overrightarrow{U_N})$;

By projection on $\overrightarrow{U_T}$ and $\overrightarrow{U_N}$ axes:

$$\begin{cases} \overrightarrow{U_T}: mg\ cos\theta = ma_T = m\dfrac{dv}{dt} \dots\dots\dots\dots\dots (1) \\ \overrightarrow{U_N}: N - mg\ sin\theta = ma_N = \ m\dfrac{v^2}{R} \dots\dots\dots (2) \end{cases}$$

We multiply both sides of equation (1) by dθ, knowing that $\frac{d\theta}{dt} = \dot{\theta} = \frac{v}{R}$, we find:

$$mg\ cos\theta d\theta = m\frac{dv}{dt}d\theta\ \Longrightarrow g\ cos\theta d\theta = \frac{d\theta}{dt}dv$$

$\Rightarrow g\ cos\theta d\theta = \frac{v}{R}dv \Rightarrow vdv = Rg\ cos\theta d\theta \ldots\ldots\ldots\ldots\ldots\ldots..(3)$

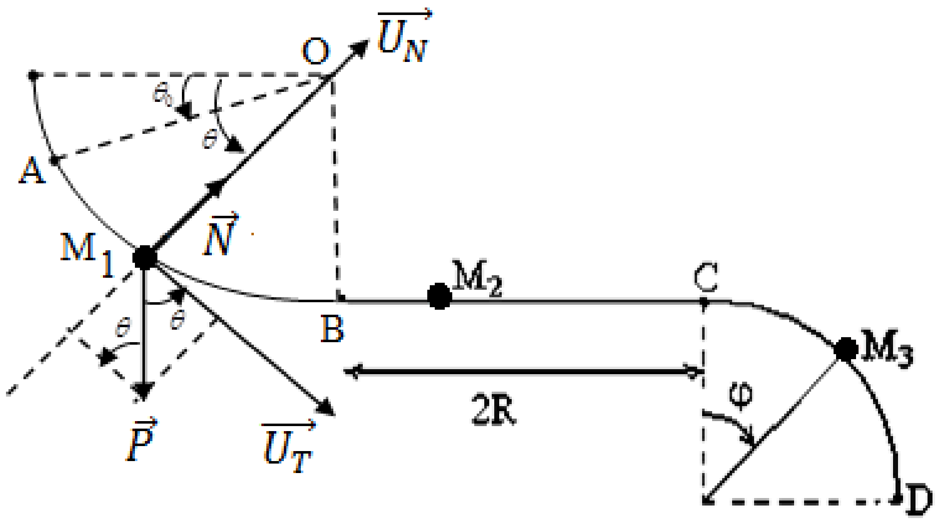
We integrate the equation (3):

$$\int_0^{v_{M1}} vdv = \int_{\theta_0}^{\theta} Rg\ cos\theta d\theta \Longrightarrow \frac{1}{2}v_{M1}^2 - 0 = Rg(sin\theta - sin\theta_0)$$

$\Rightarrow \boxed{v_{M1} = \sqrt{2Rg(sin\theta - sin\theta_0)}}$

As a vecteor, the velocity is always parallel to $\vec{U}_T$

$$\vec{v}(M_1) = \sqrt{2Rg(sin\theta - sin\theta_0)}\ \vec{U}_T$$



2/ Calculate of speed of M when it reaches the point C, considering the smooth part BC.

We apply the fundamental principle of dynamics:

$\sum \vec{F}_{ext} = m\vec{a} \Rightarrow \vec{P} + \vec{N} = m\vec{a}$

By projection on Ox and Oy axes, we find:

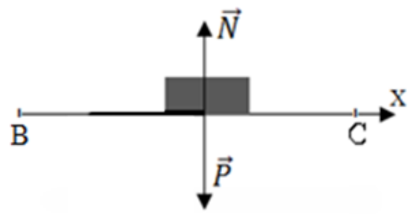
$$\begin{cases} Ox:\ 0 = m\dfrac{dv}{dt} \Longrightarrow v = cste \\ Oy: P = N \end{cases}$$

The speed is constant; the motion is rectilinear and uniform and the speed

$v_C = v_B$, we can deduce the speed at point B from the formula:

$v_{M1} = \sqrt{2Rg(\sin\theta - \sin\theta_0)}$ replacing $\theta = \pi/2$:

$$v_B = v_C = \sqrt{2Rg(1 - sin\theta_0)}$$



3/ Calculate of v_C in presence of friction on BC:

$\sum \vec{F}_{ext} = m\vec{a} \ \Rightarrow\ \vec{P} + \vec{N} + \overrightarrow{F_f} = m\vec{a}$

By projection on Ox and Oy axes, we find:

$$\begin{cases} Ox: -F_f = ma = m\dfrac{dv}{dt} \Longrightarrow -\mu N = m\dfrac{dv}{dt} \dots\dots\dots\dots.(4) \\ Oy: P = N \end{cases}$$

We multiply both sides of equation (4) by dx, knowing that $\frac{dx}{dt} = v$, we find:

$$-\mu N dx = m\frac{dv}{dt}dx = \ m\frac{dx}{dt}dv = mvdv$$

$\Rightarrow mvdv = -\mu mgdx \Longrightarrow vdv = -\mu gdx \dots\dots\dots..(5)$

We integrate the equation (5):

$$\int_{v_B}^{v_C} vdv = -\int_0^{2R} \mu gdx \Longrightarrow \frac{1}{2}v_C^2 - \frac{1}{2}v_B^2 = -2\mu gR$$

$\Rightarrow\ v_C^2 - v_B^2 = -4\mu g R \Longrightarrow v_C^2 = v_B^2 - 4\mu g R\ \ \Rightarrow \boxed{v_C = \sqrt{v_B^2 - 4\mu g R}}$

3/ Calculate of $v(M3)$ if the part CD is considered smooth:

$\sum \vec{F}_{ext} = m\vec{a}\ \ \Rightarrow\ \ \vec{P} + \vec{N} = m\vec{a}$

We use the intrinsic coordinates $(\overrightarrow{U_T}, \overrightarrow{U_N})$; by projection on theses axes:

$$\begin{cases} \overrightarrow{U_T}: mg\sin\varphi = ma_T = m\dfrac{dv}{dt} \ldots\ldots\ldots\ldots\ldots (6) \\ \overrightarrow{U_N}: mg\cos\varphi - N = ma_N = \ m\dfrac{v^2}{R} \ldots\ldots\ldots (7) \end{cases}$$

We multiply both sides of equation (6) by dx, knowing that $\frac{d\varphi}{dt} = \dot{\varphi} = \frac{v}{R}$:

$$mg\sin\varphi\, d\varphi = m\frac{dv}{dt}d\varphi\ \Longrightarrow g\sin\varphi\, d\varphi = \frac{d\varphi}{dt}dv$$

$$\Longrightarrow g\sin\varphi\, d\varphi = \frac{v}{R}dv \Longrightarrow vdv = Rg\sin\varphi\, d\varphi \ldots\ldots\ldots.\ldots.(8)$$

We integrate the equation (8):

$$\int_{v_C}^{v_{M3}} vdv = \int_0^{\varphi} Rg\sin\varphi\, d\varphi$$

$$\Longrightarrow \frac{1}{2}v_{M3}^2 - \frac{1}{2}v_C^2 = Rg(-\cos\varphi - (-\cos 0))$$

$$\Longrightarrow v_{M3}^2 - v_C^2 = 2Rg(1-\cos\varphi)$$

$$\Rightarrow \boxed{v_{M3} = \sqrt{2Rg(1 - \sin\theta_0 - 2\mu + 1 - \cos\varphi)}}$$

Calculate of the angle φ_0 for which the point M leaves the path: When M leaves the surface, the reaction force N equals zero.

From equation (7) we have:

$$N = mg\cos\varphi_0 - m\frac{v^2}{R} = 0 \Longrightarrow g\cos\varphi_0 - \frac{v^2}{R} = 0$$

On remplace la valeur de v_{M3} puisque on est sur CD:

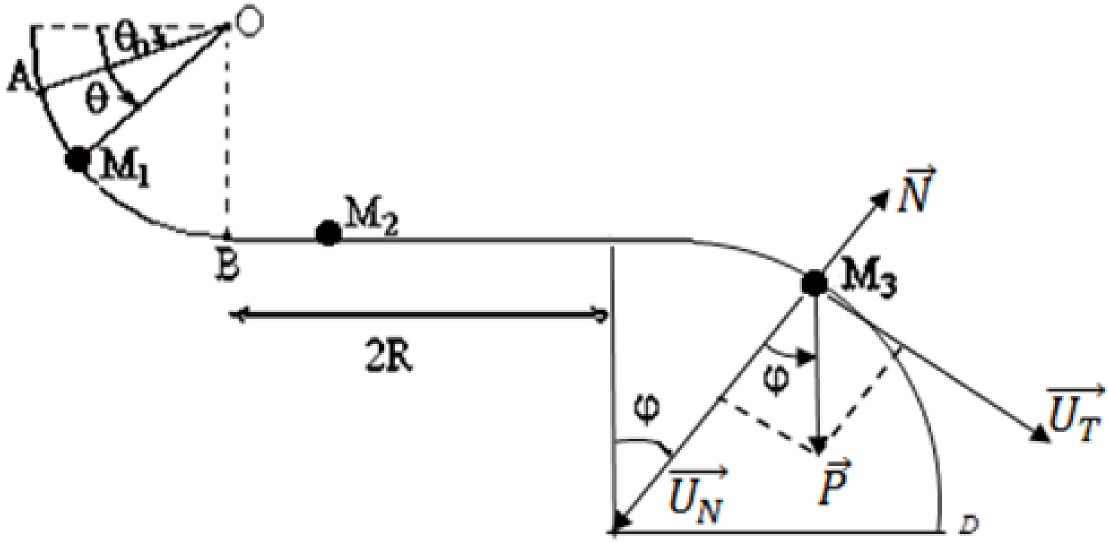
$$g\cos\varphi_0 = \frac{v_{M3}^2}{R} = \frac{2Rg(2-\sin\theta_0-\cos\varphi_0-2\mu)}{R}$$

$$\Rightarrow\ g\cos\varphi_0 = 2g(2-\sin\theta_0-\cos\varphi_0-2\mu)$$

$$\Rightarrow\ 3\cos\varphi_0 = 4-2\sin\theta_0-4\mu$$

$$\Rightarrow \cos\ \varphi_0 = \frac{4}{3}-\frac{2}{3}(\sin\theta_0+2\mu)$$

$$\Rightarrow \boxed{\varphi_0 = \arccos\left(\frac{4}{3}-\frac{2}{3}(\sin\theta_0+2\mu)\right)}$$



Set of exercises

(Dynamics of the material point)

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Exercise 1:

An object is acted on by three simultaneous forces:

$\vec{F}_1 = 2\vec{i} - 5\vec{j}\ (N)$

$\vec{F}_2 = -3\vec{i} + 3\vec{j}\ (N)$

$\vec{F}_3 = 2\vec{i} + 6\vec{j}\ (N)$

The object acceleration is equal to $1,35\ m/s^2$.

1/ Find the acceleration vector in terms of m.

2/ Find the mass of the object.

3/ If the object begins from rest, find its speed after 3 s.

4/ Find the components of the velocity of the object after 3 s.

Exercise 2:

A material point of mass m moves under the influence of a force $\vec{F}(t)$ of the form:

$$\vec{F}(t) = (2t^2 + 1)\vec{i} + 4\vec{j} + 2t\vec{k}$$

1/ Calculate the change in momentum between the two moments: $t_0 = 0s$ and $t_1 = 2s$.

2/ At t_0 the speed was:

$$\vec{v}(0) = \frac{1}{m}(-4\vec{i} + 3\vec{j} + 4\vec{k})$$

Find the velocity vector at $t_1 = 2s$.

Exercise 3:

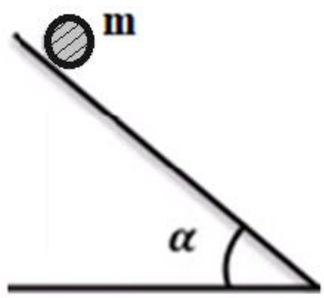
A rigid body of mass m slides over a surface inclined of an angle α from the horizon with friction coefficient μ.

1/ Plot the forces applied on the mass m.

2/ What is the relationship between the frictional force and the normal reaction force.

3/ Write Newton's second law for mass m and then find the acceleration expression in terms of μ, g and α.

Appendix I: Differential operators



Exercise 4:

An object with a mass $m = 6\ Kg$ is placed on an inclined plane that makes an angle $\alpha = 30°$. This body is subject to a horizontal force $\vec{F}$ of ($50N$) (see the figure (1)).

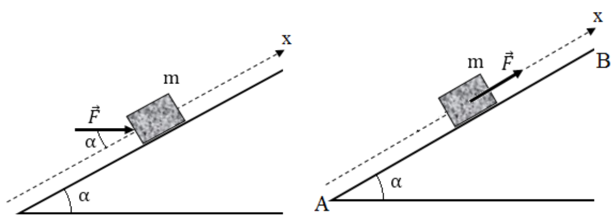


Figure (1) Figure (2)

1/ If friction is negligible, find the acceleration of the motion.

2/ If friction exists with a coefficient of friction $f = 0,15$,
Find the force of friction between the body and the inclined plane.
Find the new acceleration.

3/ We consider now that the force $\vec{F}$ is parallel to the path (see the figure (2)) and that friction is negligible.
Considering that the body is moving with an initial speed v_0 from A, find the speed of the body at point B where $AB = L$.

Exercise 5:

The position vector of a material point of mass $m = 2\ Kg$ is:

$$\vec{r} = 3t^2\vec{i} - 2t^3\vec{j} + 3t\,\vec{k}\ (m)$$

1/ Find the force applied on the body.

2/ Find the moment of $\vec{f}$ relative to the origin.

3/ Find the amount of movement of the body and its angular momentum relative to the origin (O).

4/ Check that $\vec{f} = \frac{d\vec{P}}{dt}$ and that $\overrightarrow{\mathcal{M}}_{/0} = \frac{d\vec{\mathcal{L}}}{dt}$.

Chapter IV:

Work and energy

Chapter IV: Work and energy

We have seen in the previous chapters how to solve dynamic problems using the fundamental principle of dynamics (Newton's second law). In addition to this method, we can use theorems based on kinetic, potential and mechanical energy and even the work of forces to re.

IV.1 The concept of work

The work of a force $\vec{F}$ is the energy provided by this force to move a body from one position A to another position B. It is defined as the result of the scalar product between the force and the displacement. The work W is a scalar quantity measured in Joules:

$$1\,\mathrm{J} = 1\,\mathrm{N.m} = 1\,\mathrm{Kg.m^2s^{-2}}$$

The elementary work dW of the force $\vec{F}$ at the point M which is on the line AB is equal to:

$$\mathbf{dW}(\vec{F}) = \vec{F}(M).\vec{dl}$$

$\vec{F}$ is the force applied to point M.

$\vec{dl}$ is the elementary displacement; in Cartesian coordinates, it is defined by:

$$d\vec{l} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

The total work done by a force on a body is the integral of the applied force with respect to displacement along the path AB or on a line (C) is:

$$W_{A\to B}(\vec{F}) = \int_{A/(C)}^{B} \vec{F}(M).\vec{dl}$$

Appendix I: Differential operators

If the body is subjected to several forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, $\overrightarrow{F_3}$,...,$\overrightarrow{F_n}$, whose resultant is $\overrightarrow{F_R}$, the work done by these forces is equal to the work done by the resultant force: $\overrightarrow{F_R} = \overrightarrow{F_1}+ \overrightarrow{F_2}+ \overrightarrow{F_3}+........+\overrightarrow{F_n}$.

$$W_{A\to B} = \int_A^B \vec{F}_R \,.\, d\vec{l}$$

The force can be written as: $\vec{F} \;=\; F_x\,\vec{\imath} \;+\; F_y\,\vec{\jmath} \;+\; F_z\,\vec{k}$

And the elementary displacement: $d\vec{l} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$

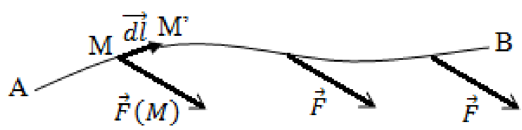
So we can write the work in analytical expression form: $dW = \vec{F}.d\vec{l} = F_x dx + F_y dy + F_z dz$

And the total work is: $W_{A\to B} = \int_A^B \vec{F}_R.d\vec{l} = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz$

If the force $\vec{F}$ is constant (in magnitude, direction, and sense; see figure below) (removing $\vec{F}$ from the integral):

$W_{A\to B}\left(\vec{F}\right) = \vec{F}\int_A^B \overrightarrow{dl} = \overrightarrow{F}.\vec{l}\,\Big|_A^B = \vec{F}\,(\vec{l}_B\text{-}\,\vec{l}_A)$

$\boxed{\boldsymbol{W_{A\to B}\left(\vec{F}\right) = \overrightarrow{F}.\overrightarrow{AB}}}$.



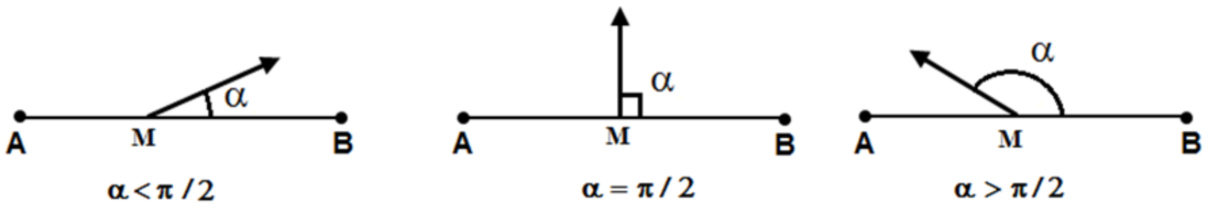
$\boxed{\boldsymbol{W_{A\to B}\left(\vec{F}\right) = \|\overrightarrow{F}\|.\|\overrightarrow{AB}\|.\cos\alpha}}$

α is the angle between the force vector $\vec{F}$ and the vector $\overrightarrow{AB}$

Assuming that the displacement AB is a straight-line segment. In this case, the values of angle α play a very important role in the contribution of the force's work:

$$\boldsymbol{dW_{A\to B}(\vec{F}) = \|\vec{F}\|.\|\overrightarrow{AB}\|.\ \cos\alpha}$$

α is the angle between the force vector $\vec{F}$ and the vector $\overrightarrow{AB}$.



The angle between the path and the force vector varies between 0° and 90°, can equal 90° or is between 90° and 180°. The nature of the Work depends on its sign. The Work can be positive; it is a driving work. It can be negative; in this case, it is a resistive work. It can be zero Work.

$\boldsymbol{\alpha < \frac{\pi}{2} \Rightarrow \cos\alpha > 0}$ **; Work is a driving force.**

$\boldsymbol{\alpha = \frac{\pi}{2} \Rightarrow \cos\alpha = 0}$ **; There is no work available.**

$\boldsymbol{\alpha > \frac{\pi}{2} \Rightarrow \cos\alpha < 0}$ **; Work is challenging.**

Example:

Calculate the work done by the force $\vec{F} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ *on a point M between the two points A* $(1, 2, 1)$ *and B* $(2, 4, 2)$.

We have: $\vec{F} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

The elementary displacement in Cartesian coordinates is: $d\,\vec{l} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$.

- $W_{A\to(B)}(\vec{F}) = \int_A^B \vec{F}\,.d\vec{l} = \int_A^B (x\vec{\imath} + y\vec{\jmath} + z\vec{k}).(dx\,\vec{\imath} + dy\vec{\jmath} + dz\vec{k})$

$= \int_A^B xdx + ydy + zd\,z$

$= \int_{xA}^{xB} xdx + \int_{yA}^{yB} ydy + \int_{zA}^{zB} zdz$

$= \frac{x^2}{2}\Big|_1^2 + \frac{y^2}{2}\Big|_2^4 \frac{z^2}{2}\Big|_1^2 = \frac{1}{2}(2^2\_1^2) + \frac{1}{2}(4^2\_2^2) + \frac{1}{2}(2^2\_2^1)$

$= \frac{2}{2}.3 + \frac{1}{2}.12 + \frac{1}{2}.3 = \frac{3}{2} + \frac{12}{2} + \frac{3}{2} = \frac{18}{2} = 9\,J$

- $W_{A\to B}(\vec{F}) = W_{A\to C}(\vec{F}) + W_{C\to B}(\vec{F}) = \int_A^C \vec{F}\,.\overrightarrow{dl} + \int_C^A \vec{F}\,.\overrightarrow{dl} = \frac{x^2}{2}\Big|_2^2 + \frac{y^2}{2}\Big|_4^4 + \frac{z^2}{2}\Big|_2^2 =$

$= \frac{1}{2}.3 + \frac{12}{2} + 0 + 0 + 0 + \frac{3}{2} = \frac{3}{2} + \frac{12}{2}\ \frac{3}{2} = 9\,J.$

The Work $W_{A\to B}(\vec{F}) = W_{A\to C}(\vec{F}) + W_{A\to B}(\vec{F}) = 9\,J.$

So, the Work W does not depend on the path taken and the force $\vec{F}$ *is conservative.*

IV.2 Power

The power of a force $\vec{F}$ is the ratio of the work of this force to the time taken to do it. We can say that that the work done by a force per unit of time is called power.

$$P = \frac{dW}{dt}$$

We can write the power as the scalar product between the force $\vec{F}$ and the velocity $\vec{v}$:

$$P = \frac{\vec{F}.d\vec{l}}{dt} = \vec{F}.\vec{v}$$

Depending on duration, the power is called average power P_{Avr} or instantaneous power P_{Ins}:

$$P_{avr} = \frac{\Delta w}{\Delta t}$$

$$P_{Ins} = \frac{dw}{dt}$$

The unit of power in the international system (IS), is the Watt (W).

$$\boxed{\mathbf{1W = 1\,J.s^{-1}}}$$

IV.3 Kinetic energy

Definition:

Consider a material point M which is in movement in a gallian frame, with a velocity $\vec{v}$, under the action of the external forces.

We have: $\sum \vec{F}_{ext} = m\,\vec{a} = \frac{d\,\vec{v}}{dt}$

The elementary Work $d\overrightarrow{W}$ of the external forces is:

Appendix I: Differential operators

$$dW_{A\to B}\left(\sum \vec{F}_{ext}\right) = \sum \vec{F}_{ext}.d\vec{l} = m.\frac{d\,\vec{v}}{dt}.d\vec{l} = m.\frac{\overrightarrow{dl}}{dt}.d\vec{v}$$

$\frac{\overrightarrow{dl}}{dt} = \vec{v}$ represents the velocity.

$\Rightarrow dW_{A\to B}\left(\sum \vec{F}_{ext}\right) = m.\vec{v}.d\vec{v} = m.v.dv$.

$W_{A\to B}\left(\sum \vec{F}_{ext}\right) = \int_A^B m\,v\,dv = \frac{1}{2}m\,v^2\,\Big|_A^B$

$\Rightarrow W_{A\to B}\left(\sum \vec{F}_{ext}\right) = \frac{1}{2}m(v_B^2 - v_A^2)$

$\Rightarrow \boxed{\boldsymbol{W_{A\to B}}\left(\sum \vec{F}_{ext}\right) = \boldsymbol{\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2}}$

We define the linear kinetic energy E_C of a material point which is in movement with a speed v in a Galilean frame by: $\boxed{\boldsymbol{E_C = \frac{1}{2}m\,v^2}}$

In the case of rotation movement, the kinetic energy takes the formula: $\boldsymbol{E_C = \frac{1}{2}J_{/\Delta}\,\dot{\theta}^2}$

Where $\dot{\theta}$ is the angular velocity and $J_{/\Delta}$ is the momentum inertia in respect to the axis of rotation Δ.

If a system of material points yields at the same time a translation and a rotation movement; the kinetic energy equals to the sum of the two energies:

$$E_C = E_{CT} + E_{CR} = \frac{1}{2}\,mv^2 + \frac{1}{2}J_{/\Delta}\,\dot{\theta}^2$$

E_{CT}: translation kinetic energy

E_{CR}: ratatina kinetic energy

Example:

Find the kinetic energy of a car of mass1000 kg moves with a velocity 100 km/h.

$$100\frac{km}{h} = 100.\frac{10^3}{3600} = 27{,}77\frac{m}{s}$$

$$E_c = \frac{1}{2}m\,v^2 = \frac{1}{2}.1000.(27{,}77)^2 = 3{,}86.10^5 = 386\ KJ$$

Theorem of kinetic energy

Enounce:

In a Galilean frame, the variation in the kinetic energy of a material point subjected to a set of external forces, between points A and B, is equal to the sum of the work of these forces between these two points:

$$\Delta E_c = E_c(B) - E_c(A) = \sum W_{A\to B}\left(\sum \vec{F}_{ext}\right)$$

IV.4 Conservative and non-conservative forces

Conservative forces

A force is called conservative when the Work done by this force does not depend on the path of movement. In the opposite case, we say the force is non-conservative.

If the path between A and B is closed, then:

$$W_{A\to B} = \oint_A^B \vec{F}.d\vec{l} = 0 \Rightarrow W_{A\to B} = 0$$

We give the example of gravitational force and the restoring force of a spring.

Gravitational force

Consider a material point of mass m, moving along a path AB. The weight of the mass m is constant.:

$$\vec{P} = -m\vec{g} = -mg\vec{k}$$

The Work of the weight trough the path AB is:

Appendix I: Differential operators

$W(\vec{P}) = \int_A^B \vec{P}.d\vec{l}$

$d\vec{l} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$

$W_{A\to B} = \int_A^B -mg\vec{k}.(dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}) = -\int_A^B mg.dz = -mg.z|_A^B = -mg(z_B - z_A)$

$\Rightarrow \boxed{W_{A\to B}(\vec{P}) = mg(z_A - z_B)}$

$h = z_A - z_B \Leftrightarrow \boxed{W_{A\to B}(\vec{P}) = mgh}$

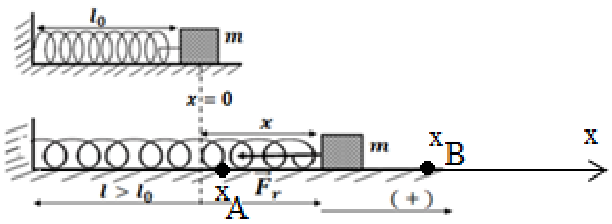
It is noted that the work done by the weight does not depend on the path AB; it only depends on the starting altitude z_A and the ending altitude z_B; in this case we say that the weight is conservative force.

Restoring force of a spring

Consider a spring with stiffness k and initial length l_0 on a horizontal plane, at the end of the spring a mass m is attached. If the mass m is moved from its initial position and released, the restoring force $\vec{F_r}$ will bring the spring to its equilibrium position. This force is given by Hooke's law:

$$\vec{F_r} = -k\vec{x}$$

k is the stiffness constant of the spring (a constant specific to that material) and x is the spring's displacement from its equilibrium length.



The Work of the restoring force of the spring is:

$W_{A\to B}(\vec{F_r}) = \int_A^B \vec{F_r}.d\vec{l}$

$\vec{F_r} = -k(l - l_0)\vec{\imath} = -kx\vec{\imath}$

$(l - l_0)$ is the displacement.

$$W_{A\to B}(\vec{F_r}) = \int_A^B -kx\vec{\imath}.(\,dx\,\vec{\imath} + \,dy\,\vec{\jmath}\; + \;dz\,\vec{k})$$

$$W_{A\to B}(\vec{F_r}) = -\int_A^B kx\,dx = -\,\frac{1}{2}kx^2|_A^B = \frac{1}{2}k(x_A^2 - x_B^2)$$

$$\boxed{\boldsymbol{W_{A\to B}(\vec{F_r}) \;=\; \frac{1}{2}\,\mathbf{k}\,(x_A^2 \;-\; x_B^2\,)}}$$

The Work of the restoring force depends only on x_A and x_B. in this case, we say, also, that the restoring force of a spring is a conservative force.

Non-conservative force

A non-conservative force is a force for which the work done depends on the path taken between two points. As a result, the work done by a non-conservative force over a closed loop is different to zero, meaning that mechanical energy is not conserved in the presence of this type of forces. Non-conservative forces typically convert mechanical energy into other forms, such as thermal or sound energy. For example, friction force, air resistance, and viscous forces.

Example

A car with a mass of $m = 800\ kg$ *is moving at a speed of* $60\ km/h$ *on a horizontal road. The driver applies the brakes, and the car comes to rest.*
What is the change in kinetic energy between the beginning and the end of braking?
How is this energy dissipated?

Solution:

The initial kinetic energy is: $E_{ci} = \frac{1}{2}m\,v^2 = \frac{1}{2}.800.\left(60.\frac{10^3}{3600}\right)^2 = 111{,}02.10^3 = 111.02\ KJ$
The final kinetic energy is: $E_{cf} = 0\ J$
$\Delta E_c = E_{cf} - E_{ci} = \dfrac{1}{2}mv_f^2 - \dfrac{1}{2}mv_i^2 = 0 - 111.02 = -111.02\ KJ$
The change in kinetic energy represents the work of conservative and non-conservative forces.
The energy is dissipated as friction and then released as heat.

IV.5 Potential energy

We define the variation of the potential energy (ΔE_p) between two points A and B by the opposite of the work of the conservative forces $\overrightarrow{F_C}$.

$$\boxed{\Delta \mathrm{E_p} = \mathrm{E_p(B)} - \mathrm{E_p(A)} = -\mathrm{W_{A\to B}}(\vec{\mathrm{F}}_c)}$$

$$\boxed{\mathrm{W_{A\to B}}(\vec{\mathrm{F}}_c) = \mathrm{E_p(A)} - \mathrm{E_p(B)}}$$

The potential energy is the energy that can be transformed in order to produce work, thus creating movement. When the variation is very small, ΔE_p becomes dE_p and, using the notion of elementary work ($\mathrm{dW} = \vec{F}.d\vec{l}$), we have:

$$\boxed{dE_p = -\overrightarrow{F_c}.d\vec{l}}$$

In Cartesian coordinates, the differential of a function $f(x,y,z)$ is defined by:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

So the differential of the potential energy $E_P(x,y,z)$ is:

$$dE_P = \frac{\partial E_p}{\partial x}dx + \frac{\partial E_p}{\partial y}dy + \frac{\partial E_p}{\partial z}dz$$

$$dE_p = -\overrightarrow{F_c}.\mathrm{d}\vec{\mathrm{l}}$$

$$\overrightarrow{F_c} = F_{cx}\vec{\imath} + F_{cy}\vec{\jmath} + F_{cy}\vec{k};\ d\vec{l} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

$$dE_P = -F_{cx}d_x - F_{cy}d_y - F_{cz}d_z$$

By comparison: $\begin{cases} \frac{\partial E_p}{\partial x} = -F_{cx} \\ \frac{\partial E_p}{\partial y} = -F_{cy} \\ \frac{\partial E_p}{\partial z} = -F_{cz} \end{cases}$

We can write $\overrightarrow{F_c}$ in the form:

Appendix I: Differential operators

$$\boxed{\overrightarrow{F_c} = -\frac{\partial E_p}{\partial x}\vec{\imath} - \frac{\partial E_p}{\partial y}\vec{\jmath} - \frac{\partial E_p}{\partial z}\vec{k}}$$

It is said that $\overrightarrow{F_c}$ is derived from the potential E_p.

On the other hand, let the gradient operator ($\overrightarrow{grad}$) of a function f be defined by:

$$\overrightarrow{grad}f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}$$

We define a point M, identified in the Oxyz coordinate system by its position vector $\overrightarrow{OM}$, such that:

$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\overrightarrow{k} \Rightarrow d\overrightarrow{OM} = dx\vec{\imath} + dy\vec{\jmath} + dz\overrightarrow{k}$

$\overrightarrow{grad}f.d\overrightarrow{OM} = \left(\frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}\right).\left(dx\vec{\imath} + dy\,\vec{\jmath} + dz\overrightarrow{k}\right) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

So; $\boxed{\overrightarrow{grad}f.d\overrightarrow{OM} = df}$

And we have: $dE_p = -\overrightarrow{F_c}.d\vec{l}$

We can note that: $\overrightarrow{grad}E_p.d\vec{l} = dE_p = -\overrightarrow{F_c}.d\vec{l}$

We can finally write the relationship that links the conservative force $\overrightarrow{F_c}$ and the potential E_p from which the force was derived:

$$\boxed{\boldsymbol{\overrightarrow{F_c} = -\overrightarrow{grad}E_p}}$$

For example, of conservative forces, we found gravitational force: $\overrightarrow{F_C} = \vec{P} = -mg\vec{k}$

$\overrightarrow{F_C} = -\overrightarrow{grad}E_p \Rightarrow -mg\vec{k} = -\frac{dE_P}{dz}\vec{k} \Rightarrow mg = \frac{dE_P}{dz}$

$\Rightarrow \mathrm{dE_P = mgdz} \Rightarrow \int \mathrm{dE_P} = \int \mathrm{mgdz} \Rightarrow \boxed{\mathrm{E_P = mgz + cste}}$

E_p (or E_{pp}) is a type of potential energy known as gravitational energy.

z is the height at which the point M is located relative to a reference level. If we choose a reference for potential energy $E_P = 0$, the value of z is positive if the moving body is above the reference position, and if the body is below this reference, the value of z is negative.

The second example of conservative forces we found elastic force (Restoring force of a spring):

$$\overrightarrow{F_C} = \overrightarrow{F_r} = -kx\vec{\imath}$$

$$\overrightarrow{F_C} = -\overrightarrow{grad}E_p \Rightarrow -kx\vec{\imath} = -\overrightarrow{grad}E_p = -\frac{dE_P}{dx}\vec{\imath}$$

$$\Rightarrow kx = \frac{dE_P}{dx} \Rightarrow \int dE_P = \int kxdx$$

$$\Rightarrow \boxed{\boldsymbol{E_P = \frac{1}{2}kx^2 + cste}}$$

E_p (or E_{pe}) is elastic potential energy.

Equilibrium condition

The condition of equilibrium according to Newton's first law is: $\Sigma\vec{F}_{ext} = \vec{o}$

If the material system is subjected to the sum of conservative forces, then: $\Sigma\vec{F}_{ext} = \vec{F_c} = -\overrightarrow{grad}E_p$

So, the equilibrium condition is: $-\overrightarrow{grad}E_p = \vec{0}$

$$\Rightarrow -\frac{\partial E_p}{\partial x}\vec{\imath} - \frac{\partial E_p}{\partial y}j - \frac{\partial E_p}{\partial z}\vec{k} = \vec{0}$$

$$\Rightarrow \begin{cases} \frac{\partial E_p}{\partial x} = 0 \\ \frac{\partial E_p}{\partial y} = 0 \\ \frac{\partial E_p}{\partial z} = 0 \end{cases}$$

Stable equilibrium

If the force exerted on the material system is a restoring force, i.e. the force always pushes the body towards equilibrium, and the direction of movement is always opposite to the direction of the force, the equilibrium is said to be stable. The body tilts around a position of equilibrium.

Instable equilibrium

It is characterized by a force that constantly pushes the body away from its position of equilibrium. The direction of movement is the same as the direction of the force, so the body will move away from its position of equilibrium.

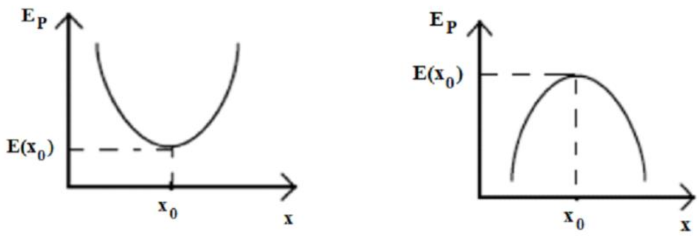
To find the equilibrium points, you need to calculate $\begin{cases} \frac{\partial E_p}{\partial x} = 0 \\ \frac{\partial E_p}{\partial y} = 0 \\ \frac{\partial E_p}{\partial z} = 0 \end{cases}$

In one dimension $E_p = f\ (x)$, the points that cancel out E_p are the equilibrium points.

Let $x_1$and x_2 be equilibrium points. To determine the type of equilibrium, we must calculate

$\frac{\partial^2 E_p}{\partial x}$ for $x_1$and x_2.

If: $\frac{\partial^2 E_p}{\partial x^2}|_{x=x_1} > 0 \Leftrightarrow$ We have stable equilibrium and if: $\frac{\partial^2 E_p}{\partial x^2}|_{x=x_2} < 0\ \Leftrightarrow$ instable equilibrium.



Stable equilibrium   Instable equilibrium

Example

Let the potential U, with: $U = 2x^3 - x^2$.
Find the equilibrium points and specify the type of equilibrium.

Solution:

$$\frac{\partial U}{\partial x} = 6x^2 - 2x$$

To find the equilibrium points, one must calculate: $\frac{\partial U}{\partial x} = 0$

$$\Rightarrow 6x^2 - 2x = 0$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ \quad or \\ x_2 = \frac{1}{3} \end{cases}$$

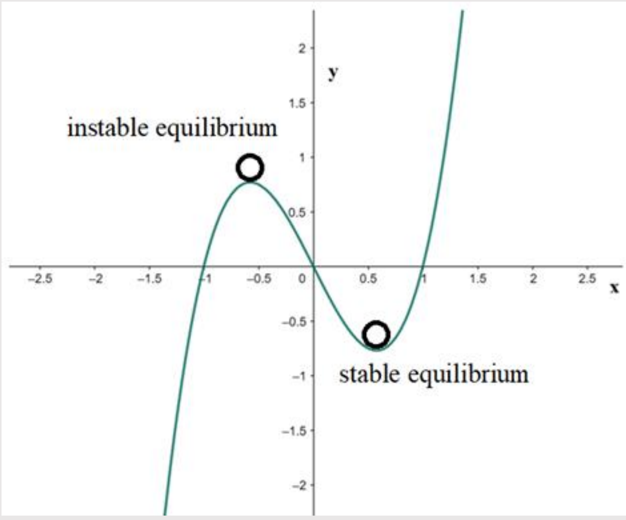
x_1 *et* x_2. are the equilibrium points and to determine the type of equilibrium, we should calculate the second derivative of the potential at these two points: $\frac{\partial^2 U}{\partial x^2} = 12x - 2.$

$$\frac{\partial^2 U}{\partial x^2}\Big|_{x_1} = 12.(0) - 2 = -2$$

$\frac{\partial^2 U}{\partial x^2}\Big|_{x_1} < 0 \Leftrightarrow x = 0$ is a stable equilibrium.

$$\frac{\partial^2 U}{\partial x^2}\Big|_{x_2} = 12.\left(\frac{1}{3}\right) - 2 = 4 - 2 = 2 > 0$$

$\frac{\partial^2 U}{\partial x^2}\Big|_{x_2} > 0 \Leftrightarrow x = \frac{1}{3}$ is an instable equilibrium.



IV.6 Mechanical energy

The mechanical energy of a material point is equal to the sum of the kinetic energy and the potential energy:

Appendix I: Differential operators

$$\boxed{\boldsymbol{E_M = E_c + E_p}}$$

For the mass-spring system (m-k), the kinetic energy and potential energy are: $E_c = \frac{1}{2}mv^2$, $E_p = \frac{1}{2}kx^2$

The mechanical energy is the sum of the two energies: $E_M = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$.

Mechanical energy and non-conservative forces

Consider a system subjected to conservative forces and non-conservative forces; the work of these forces between points A and B is equal to: $W_{A\to B} = W_{A\to B}(\vec{F_c}) + W_{A\to B}\ (\vec{F}_{Nc})$

$W(\vec{F_c})$ is the Work of conservative forces and $W(\vec{F}_{Nc})$ is the Work of non-conservative forces.

According to the kinetic energy theorem: $E_c(B) - E_c(A) = \sum W_{A\to B}(\vec{F}_{ext})$

$\vec{F}_{ext}$ represents the sum of $\vec{F_c}$ and $\vec{F}_{Nc}$.

And we note that the variation in potential energy is equal to: $E_p(B) - E_p(A) = -W_{A\to B}(\vec{F_c})$

$\Rightarrow W_{A\to B}(\vec{F_c}) = E_p(A) - E_p(B)$

So: $\sum W_{A\to B} = E_c(B) - E_c(A) = W_{A\to B}(\vec{F_c}) + W_{A\to B}(\vec{F}_{Nc})$.

$\Rightarrow E_c(B) - E_c(A) = E_p(A) - E_p(B) + W_{A\to B}(\vec{F}_{Nc})$.

$\Rightarrow E_c(B) + E_p(B) - E_c(A) - E_p(A) = W_{A\to B}(\vec{F}_{Nc})$

$\Rightarrow \left(E_c(B) + E_p(B)\right) - \left(E_c(A) + E_p(A)\right) = W_{A\to B}(\vec{F}_{Nc})$

$\Rightarrow E_M(B) - E_M(A) = W_{A\to B}(\vec{F}_{Nc})$

So: $\boxed{\Delta \boldsymbol{E_M = W_{A\to B}(\vec{F}_{Nc})}}$

It can be said that the work of non-conservative forces represents the dissipation of energy.

Mechanical energy theorem

The variation in the mechanical energy of a body between point A and point B is equal to the sum of work of the non-conservative forces acting on this body:

$$\boxed{E_M(B) - E_M(A) = W_{A\to B}\left(\vec{F}_{Nc}\right)}$$

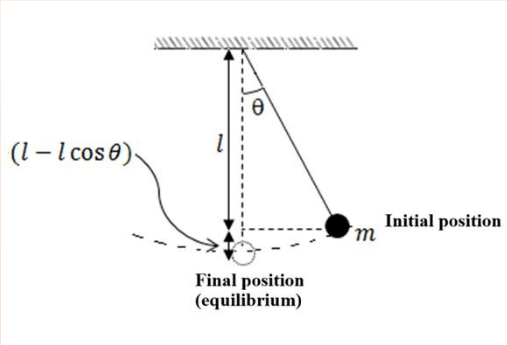
In the case where all forces are conservative, the change in mechanical energy is therefore zero: $\boldsymbol{\Delta E_M = 0}$ the mechanical energy is therefore conserved: $E_M(B) = E_M(A)$

In other words, if the system is mechanically isolated, mechanical energy is conserved.

We use the theorem of conservation of mechanical energy $\Delta E_M = 0$ to solve mechanical problems by deriving the formula for this energy in relation to time, since this energy is always constant.

Application:

Consider a simple pendulum (string length $l = 1\,m$ and mass $m = 200\,g$). The mass m is released from position $\theta = 15°$ with zero initial velocity. Friction is negligible.
Use the mechanical energy theorem to calculate:
1/ The velocity v_e when passing through the vertical position (equilibrium position $\theta = 0°$).
2/ The period of oscillation of the pendulum.



Solution:

Friction is negligible, meaning that mechanical energy is conserved: $\Delta E_M = 0$; so: $E_M(i) = E_M(e)$
$E_M(i)$ is initial mechanical energy at initial position: $\theta = 15°$
$E_M(e)$ is final mechanical energy (equilibrium position) at final position: $\theta = 0°$
$E_c(i) + E_{pp}(i) = E_c(e) + E_{pp}(e)$
E_c linear kinetic energy and E_{pp} gravitational potential energy.

$$\frac{1}{2}mv_e^2 + mgh_e = \frac{1}{2}mv_i^2 + mgh_i$$

If we take $\theta = 0°$ as reference of potential energies $E_p = 0 \Leftrightarrow h_e = 0$ and $h_i = l - l\cos\theta$ (hi is positive)

$$\frac{1}{2}mv_e^2 + mg0 = \frac{1}{2}m\,0^2 + mg(l - l\cos\theta)$$

$$\Rightarrow v_e = \sqrt{2gl(1-\cos\theta)} = \sqrt{2.10.1(1-\cos 15)} = 0{,}82\ m/s$$

If we take $\theta = 15°$ as reference of potential energies $E_p = 0 \Leftrightarrow h_i = 0$ and $h_e = -(l - l\cos\theta)$ (h_e is negative; below $E_p = 0$).

$$\frac{1}{2}mv_e^2 - mg(l - l\cos\theta) = \frac{1}{2}m\,0^2 + 0 \Rightarrow v_e = \sqrt{2gl(1-\cos\theta)} = 0{,}82\ m/s$$

Calculation of the period: The mechanical energy is conserved, i.e. it does not depend on time $\Rightarrow \frac{d\Delta E_M}{dt} = 0$
At a given instant t with velocity v:

$$\Delta E_M = E_M(i) - E_M(t) = \frac{1}{2}mv_i^2 + mgh_i - \frac{1}{2}mv^2 - mgh = -\frac{1}{2}mv^2 - mgl(1-\cos\theta)$$

$$\frac{d\Delta E_M}{dt} = 0 \Leftrightarrow \frac{d}{dt}\left(-\frac{1}{2}mv^2 - mgl(1-\cos\theta)\right) = 0$$

We have $v = R\dot{\theta} = l\dot{\theta}$; $R = l$ is the radius of rotation of mass m, and we have the limited expansion of order 2 of $\cos\theta$ is: $\cos\theta = 1 - \frac{\theta^2}{2}$

$$So: \Delta E_M = -\frac{1}{2}m\left(l\dot{\theta}\right)^2 - mgl\left(1 - 1 + \frac{\theta^2}{2}\right) = -\frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\,\theta^2$$

$\frac{d\Delta E_M}{dt} = 0 \Leftrightarrow -\frac{1}{2}ml^2.2\dot{\theta}\ddot{\theta} - \frac{1}{2}mgl.2\theta\dot{\theta} = 0$; Because: $\frac{d\dot{\theta}^2}{dt} = \frac{d\dot{\theta}.\dot{\theta}}{dt} = \dot{\theta}.\frac{d\dot{\theta}}{dt} + \dot{\theta}.\frac{d\dot{\theta}}{dt} = \dot{\theta}\ddot{\theta} + \dot{\theta}\ddot{\theta} = 2\dot{\theta}\ddot{\theta}$ and $\frac{d\theta^2}{dt} = 2\theta\dot{\theta} \Rightarrow \dot{\theta}m\left(l\ddot{\theta} + g\theta\right) = 0$; $\dot{\theta} \neq 0$ so: $l\ddot{\theta} + g\theta = 0 \Rightarrow \boxed{\ddot{\boldsymbol{\theta}} + \frac{\boldsymbol{g}}{\boldsymbol{l}}\boldsymbol{\theta} = \mathbf{0}}$

It is a differential equation of the form $\ddot{\theta} + \omega_0^2\theta = 0$. It represents the sinusoidal motion equation with frequency $\omega_0 = \sqrt{\frac{g}{l}}$, and we have $\omega_0 = \frac{2\pi}{T}$; $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}} = 2\pi.\sqrt{\frac{1}{10}} = 1{,}98\ s$

Set of exercises

(Work and energy)

Set of exercises (Work and energy)

Exercise 1:

A material particle M moves in the plane xOy under the action of force $\vec{F}$ with:

$$\vec{F} = (y^2 - x^2)\vec{\imath} + 3xy\vec{\jmath}$$

Consider the points: $O(0,0), A(2,0), B(0,2)$ and $C(2,2)$

Find the work done by force $\vec{F}$ when point M moves from O to C trough:

1/ Along the straight segment OC.

2/ Along the path $O \to A \to C$.

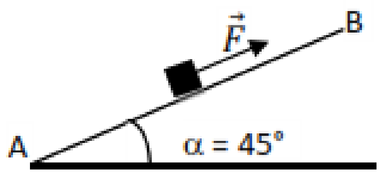
3/ Along the path $O \to B \to C$.

- What is the nature of force $\vec{F}$?

Exercise 2:

Calculate the work done by the forces applied to cube M of mass m along the path $AB = 10\ m$.

$m\ = 1\ kg, a = 1\ m/s^2$



Exercise 3:

A material point m slides from point O with no initial velocity on a plane inclined at an angle α (figure). We assume that there is friction on the OA section and that the coefficient of kinetic friction is μ, and that there is no friction on the AB section. On this plane, a spring is attached to point B with length L_0 and stiffness constant k.

1/ Calculate the friction force as a function of α, μ, m and g, where g is the acceleration due to gravity.

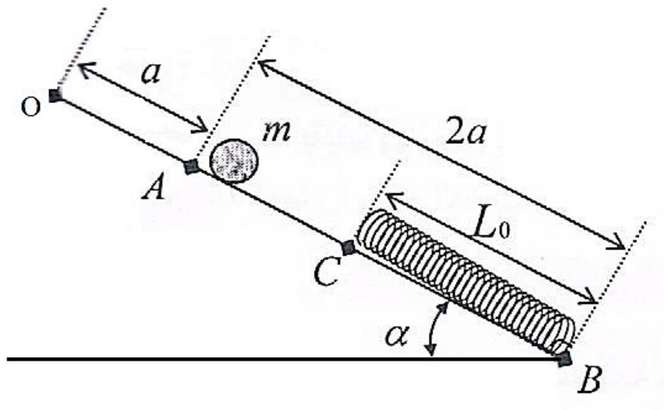
2/ Calculate the velocity at point A using:

a- Kinetic energy theorem.

b- Mechanical energy theorem.

3/ Apply the kinetic energy theorem to calculate the velocity of m at point C.

4/ What is the maximum compression d of the spring resulting from the mass m?



N.A: $m = 5\,kg$; $\alpha = 60\,°$; $L_0 = 40\,cm$; $k = 5000\,N.m^{-1}$; $\mu = 0{,}2$; $a = 1\,m$;

$g = 9{,}8\,m.s^{-2}$

Appendix I: Differential operators

Appendix I: Differential operators

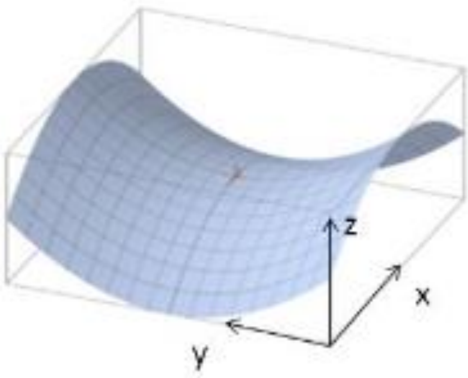
I.1 Scalar field

If we can assign to each point $M(x, y, z)$ in space a scalar quantity $f(x, y, x)$, the set of quantities of the functions $f(M)$ represents a scalar field. We give the example of temperature values in a room; each value is different from the other.

Examples:

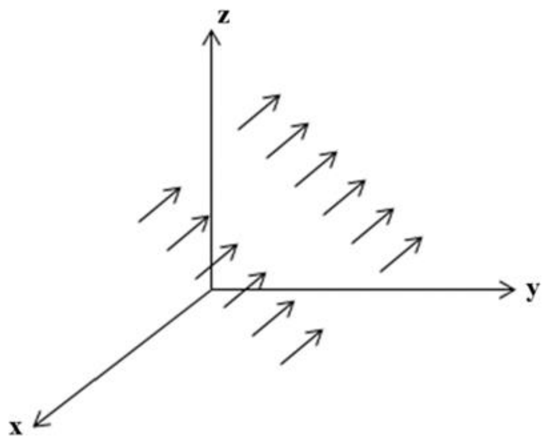
1/ $f(x, y, z) = x^2 + y^2 + z^2$ is a function for points in space $M(x, y, z)$, we can define the scalar field by the set of $f(x, y, z)$, these values are scalar quantities.

2/ The function f(x,y) given by $(x, y) = x^2 + y^2$ is a function of two variables x and y, the graphical representation is shown in the figure.



I.2 Vector field

If we can assign to each point in space a vector quantity $\vec{F}(x, y, z)$, the set of vector quantities in space represents a vector field.



Example:

The set of vectors of the scalar function $\vec{F}(x,y,z)$ represents a vector field.

$$\vec{F}(x,y,z) = (6x^2 - z)\vec{\imath} + (y - 2x)\vec{\jmath} + (3z - xy)\overrightarrow{k}$$

I.3 The flow and circulation of a vector field

The flow Φ of a vector field through a surface (S) is given by the relation:

$$\boxed{\Phi(\vec{E}) = \iint_S \vec{E}.\overrightarrow{dS} = \iint_S \vec{E}.dS.\vec{n}}$$

Where $\vec{n}$ is the unit vector perpendicular to the surface S.

The circulation of a vector field on a path (Γ) is given by the relation:

$$\boxed{C(\vec{E}) = \int_\Gamma \vec{E}.\overrightarrow{dl}}$$

I.4 The operators

Operators are mathematical tools that influence scalar fields and vector fields to give another field (scalar or vector).

Nabla operator ∇

In Cartesian coordinates we define the operator nabla ∇ by the vector:

Appendix I: Differential operators

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k} \; ; \vec{\nabla} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

The relationships of Nabla in cylindrical and spherical coordinates are:

In cylindrical coordinates:

$$\vec{\nabla} = \frac{\partial}{\partial r}\overrightarrow{U_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\overrightarrow{U_\theta} + \frac{\partial}{\partial z}\vec{k} \; ;$$

In spherical coordinates:

$$\vec{\nabla} = \frac{\partial}{\partial r}\overrightarrow{U_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\overrightarrow{U_\theta} + \frac{1}{r \sin\varphi}\frac{\partial}{\partial \varphi}\overrightarrow{U_\varphi}$$

Gradient operator

The gradient is the influence of the Nabla operator on a scalar field, the gradient describes a vector field:

$$\boxed{\overrightarrow{grad}\, V = \vec{\nabla}.V = \frac{\partial V}{\partial x}\vec{\imath} + \frac{\partial V}{\partial y}\vec{\jmath} + \frac{\partial V}{\partial z}\vec{k}}$$

We define the differential dV of a scalar function V(x,y,z) by:

$$\boxed{dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz}$$

The integral of the differential of a function between two points A and B gives the difference between the values of V(B) minus V(A):

$$\boxed{\int_A^B dV = V\Big|_A^B = V(B) - V(A)}$$

Divergence operator

It is the influence of the Nabla operator on a vector field:

($\vec{E} = E_x\vec{\imath} + E_y\vec{\jmath} + E_z\vec{k}$) by a scalar product, the divergence is therefore a scalar:

$$div\,\vec{E} = \vec{\nabla}.\vec{E} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right).\left(E_x\vec{\imath} + E_y\vec{\jmath} + E_z\vec{k}\right) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\boxed{div\,\vec{E} = \vec{\nabla}.\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}$$

Rotational operator

It is the influence of the Nabla operator on a vector field $\vec{E} = E_x\vec{\imath} + E_y\vec{\jmath} + E_z\vec{k}$ by a vector product, the rotational is therefore a vector:

$$\overrightarrow{rot}\vec{E} = \vec{\nabla}\wedge\vec{E} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right)\wedge\left(E_x\vec{\imath} + E_y\vec{\jmath} + E_z\vec{k}\right) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\boxed{\overrightarrow{rot}\vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\vec{\imath} - \left(\frac{\partial E_z}{\partial x} - \frac{\partial}{\partial z}E_x\right)\vec{\jmath} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\vec{k}}$$

If the rotational of a vector is zero ($\overrightarrow{rot}\vec{E} = \vec{0}$) we say that the vector $\vec{E} = E_x\vec{\imath} + E_y\vec{\jmath} + E_z\vec{k}$ is derived from a potential V, this potential is the scalar function $V(x, y, z)$: $\boxed{\vec{E} = -\overrightarrow{grad}V}$

Scalar Laplacian operator

$$\Delta V = div(\overrightarrow{grad}V)$$

$$\Delta V = \vec{\nabla}.\vec{\nabla}\,V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Vector Laplacian operator

$$\Delta\vec{E} = \vec{\nabla}.\vec{\nabla}\,\vec{E} = \Delta E_x\vec{\imath} + \Delta E_y\vec{\jmath} + \Delta E_z\vec{k} = \vec{\nabla}.\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right).\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\vec{\imath} + \frac{\partial}{\partial y}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\vec{\jmath}$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\vec{k}$$

I.5 Operational analysis formulas

$$\overrightarrow{rot}\left(\overrightarrow{grad}V\right) = \vec{0}$$

$$div\,(\overrightarrow{rot}\vec{E}) = 0$$

$$\overrightarrow{rot}\left(\overrightarrow{rot}\vec{E}\right) = \overrightarrow{grad}\left(div\vec{E}\right) - \Delta\vec{E}$$

$$div\left(V.\vec{E}\right) = \vec{E}.\overrightarrow{grad}V + V\,div\vec{E}$$

$$\overrightarrow{rot}\left(V\vec{E}\right) = V\overrightarrow{rot}\vec{E} - \vec{E}\wedge\overrightarrow{grad}V$$

$$\overrightarrow{grad}(f.g) = g\left(\overrightarrow{grad}f\right) + f(\overrightarrow{grad}g)$$

$$div\left(\vec{A}\wedge\vec{B}\right) = \vec{B}.\overrightarrow{rot}\vec{A} - \vec{A}.\overrightarrow{rot}\vec{B}$$

$$\overrightarrow{rot}\left(\vec{A}\wedge\vec{B}\right) = \vec{A}\,div\vec{B} - \vec{B}\,div\,\vec{A} + \left(\vec{B}.\overrightarrow{grad}\right)\vec{A} + (\vec{A}.\overrightarrow{grad})\vec{B}$$

$$\overrightarrow{grad}\left(\vec{A}.\vec{A}\right) = 2\vec{A}\wedge\overrightarrow{rot}\vec{A} + 2(\vec{A}.\overrightarrow{grad})\vec{A}$$

$$\Delta(f.g) = f\Delta g + g\Delta f + 2\overrightarrow{grad}f.\overrightarrow{grad}g$$

Appendix II: Terminological Equivalents

Appendix II: Terminological Equivalents

N°	Word in English	Mot en Français	الكلمة بالعربية
1	Absolute	Absolu	مطلق
2	Acceleration	Accélération	تسارع
3	Acted upon to	Soumise	يخضع
4	Along	Le long	على امتداد
5	Amount	Quantité	كمية
6	Angle	Angle	زاوية
7	Angular	Angulaire	زاوي
8	Angular momentum	Moment angulaire	الزخم الزاوي
9	Appendix	Annexe	ملحق
10	Applied	Appliqué	مطبق
11	Appropriate	Approprié	ملائم
12	Arrow	Flèche	سهم
13	Assume	Supposer	يفترض
14	Axis	Axe	محور
15	Basis	Base	قاعدة
16	Bottom	Bas, fond	أسفل
17	Cartesian coordinates	Coordonnées cartésiennes	لإحداثيات الديكارتية
18	Center of the circle	Centre du cercle	مركز الدائرة
19	Centripetal	Centripète	مركزي
20	Centripetal acceleration	Accélération centripète	تسارع مركزي
21	Challenging	Difficile	صعب
22	Circle	Cercle	دائرة
23	Circular motion	Mouvement circulaire	الحركة الدائرية
24	Circulation	Circulation	تجوال

25	Clockwise	Dans le sens des aiguilles d'une montre	في اتجاه عقارب الساعة
26	Coefficient	Coefficient	عامل
27	Committee	Comité	لجنة
28	Commutativity	Commutativité	تبديلي
29	Component	Composant	مركبة
30	Composition	Composition	تركيب
31	Concept	Concept	مفهوم
32	Concurrent	Concourantes	ملتقية
33	Conservative	Conservateur	محافظ
34	Counterclockwise	Sens inverse des aiguilles d'une montre	عكس عقارب الساعة
35	Curve	Courbe	منحنى
36	Curvilinear	Curvilinéaire , curviligne	منحني
37	Cylindrical coordinates	Coordonnées cylindriques	الإحداثيات الأسطوانية
38	Derivative	Dérivée	اشتقاق
39	Derive	Dérivé	اشتق
40	Describe	Décrire	يصف
41	Differential	Différentiel	تفاضل
42	Dimension	Dimension	البعد
43	Direction	Direction	اتجاه
44	Displacement	Déplacement	الإزاحة
45	Distance	Distance	مسافة
46	Driving	Conduite	قيادة
47	Dynamic	Dynamique	تحريك
48	e.g.	Exemple	مثل
49	Elastic	Elastique	مرن
50	Ellipse	Ellipse	قطع ناقص
51	Energy	Energy	طاقة
52	Equation	Equation	معادلة

53	Equilateral triangle	Triangle équilatéral	مثلث متساوي الأضلاع
54	Equilibrium	Equilibre	توازن
55	Equivalent	Equivalent	مكافئ
56	Etc.	Etcetera	الى أخره
57	Extension	Extension	استطالة
58	Flow	Flux	تدفق
59	Force	Force	قوة
60	Formulas	Formules	صيغ
61	Frequency	Fréquence	تواتر
62	Friction	Frottement	احتكاك
63	Function	Fonction	دالة
64	Gravitational	Gravitationnel	جاذبية
65	Horizontal	Horizontale	أفقي
66	However	Alors que	بينما
67	i.e.	C'est-à-dire	أي
68	Immobile	Immobile	غير متحرك
69	Include	Inclure	يشمل
70	Increases	Augmente	يزيد
71	Inertia	Inertie	قصور الذاتي
72	Initial phase	Phase initiale	الطور الابتدائي
73	Instantaneous	Instantanée	لحظي
74	Interval	Intervalle	فاصل
75	Intrinsic	Intrinsèque	ذاتي
76	Isosceles	Isocèle	متساوي الساقين
77	Kinematics	Cinématique	حركيات
78	Law	Loi	قانون
79	Left	Gauche	يسار
80	Length	Longueur	الطول
81	Lie	Se trouvent	يقع

82	Luminous intensity	Intensité lumineuse	شدة الإضاءة
83	Magnitude	Magnitude	الطويلة
84	Material point	Point matériel	نقطة مادية
85	Mechanical	Mécanique	ميكانيكي
86	Metric	Métrique	قياس
87	Momentum	Moment	زخم
88	Motion	Mouvement	حركة
89	Multitude	Multitude	العديد
90	Natural	Naturel	طبيعي
91	Negligible	Négligeable	مهمل
92	One-dimensional	Unidimensionnel	بعد واحد
93	Operator	Opérateur	مؤثر
94	Oriented	Orienté	موجه
95	Orthogonal	Orthogonal	متعامد
96	Orthoradial	Orthoradial	عمودي على القطر
97	Parabola	Parabole	قطع مكافئ
98	Parallelepiped	Parallélépipède	متوازي السطوح
99	Parallelogram	Parallélogramme	متوازي أضلاع
100	Particular	Particulier	خاص
101	Pendulum	Pendule	نواس
102	Period	Période	دور
103	Periodic	Périodique	دوري
104	Plane	Plan	مستوي

105	Polar	Polaire	القطبية
106	Polar coordinates	Coordonnées polaires	لإحداثيات القطبية
107	Position	Position	موضع
108	Power	Puissance	استطاعة
109	Proportional	Proportionnel	تناسب
110	Quantity	Grandeur	مقاس
111	Radial	Radiale	نصف قطري
112	Radius	Rayon	نصف القطر
113	Rectilinear	Rectiligne	مستقيمة
114	Reference	Référence	مرجع
115	Refers	Se réfère	يشير
116	Relative	Relative	نسبي
117	Reminder	Rappel	تذكير
118	Restoring	Restaurer	إعادة
119	Right	Droite	يمين
120	Right triangle	Triangle rectangle	مثلث قائم
121	Rotational	Rotationnel	دورانية

122	Scalar	Scalaire	عددي
123	Second derivative	Dérivée seconde	المشتق الثاني
124	Segment	Segment	قطعة مستقيمة
125	Set	Ensemble	مجموعة
126	Side	Coté	جانب
127	Sinusoidal	Sinusoïdal	جيبي
128	So	Donc	إذن
129	Speed	Vitesse	سرعة
130	Spherical coordinates	Coordonnées sphériques	الإحداثيات الكراوية
131	Spring	Ressort	نابض
132	Stable	Stable	مستقر
133	Stiffness	Raideur	صلابة
134	Straight line	Ligne droite	خط مستقيم
135	Straight motion	Mouvement rectiligne	الحركة المستقيمة
136	Substituting	Remplacement	استبدال
137	Supplement	Supplément	إضافي
138	Temperature	Température	درجة الحرارة

139	Tension	Tension	توتر
140	Terminological	Terminologique	مصطلحي
141	Three-dimensional	Tridimensionnel	ثلاث أبعاد
142	Time	Temps	زمن
143	Top	Sommet	أعلى، قمة
144	Training	Entraînement	جر
145	Trajectory	Trajectoire	مسار
146	Transmit	Transmettre	نقل
147	Triangle	Triangle	مثلث
148	Trigonometric functions	Fonctions trigonométrique	الدوال المثلثية
149	Two-dimensional	Bidimensionnel	بعدين
150	Unit	Unité	وحدة
151	Vector	Vecteur	شعاع
152	Velocity	Vitesse	سرعة
153	Vertical	Verticale	عمودي، رَأسِيّ
154	Viscous	Visqueux	لزج

155	Weight	Poids	ثقل
156	With respect to	Par rapport	بالنسبة إلى
158	Work	Travail	عمل

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